The Irrelevance of Market Incompleteness for the Price of Aggregate Risk*

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Abstract

This paper shows that in a class of incomplete markets model with large number of agents that have constant relative risk aversion (CRRA) preferences the lack of insurance markets for idiosyncratic labor income risk has no effect on the premium for aggregate risk, *if* the distribution of idiosyncratic risk is independent of aggregate shocks. Despite missing markets, the representative agent prices the excess returns on stocks correctly. As a result, in this class of models there is no link between the extent of self-insurance against idiosyncratic income risk and risk premia. This result holds regardless of the persistence of the idiosyncratic shocks (as long as they are not permanent) and is true even when households face occasionally binding, and potentially very tight borrowing constraints.

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1 Introduction

This paper shows that in a large class of incomplete markets models there is no link between the extent to which idiosyncratic risk is traded away by households and the size of the risk premium that stocks command over risk-free bonds in financial markets. In a standard incomplete markets model populated by a continuum of agents who have CRRA preferences and can only trade a risk-free bond, the presence of uninsurable idiosyncratic risk only affects the equilibrium risk-free rate, but it has no effect on the premium for aggregate risk in equilibrium if the distribution of idiosyncratic shocks is statistically independent of aggregate shocks. Consequently, in this class of models the representative agent Consumption-CAPM (CCAPM) developed by Breeden (1979) and Lucas (1978) prices the excess returns on the stock correctly. Therefore, as long as idiosyncratic shocks are independent of aggregate shocks, the extent to which households manage to insure against idiosyncratic income risk is irrelevant for risk premia. These results deepen the equity premium puzzle, because we show that Mehra and Prescott's (1985) statement of the puzzle applies to a much larger class of incomplete market models.[1]

Our result holds regardless of the persistence of the idiosyncratic shocks, as long as these shocks are not completely permanent, and it is robust to the introduction of various borrowing and solvency constraints, regardless of the tightness of these constraints, as long as these constraints allow for the existence of a stationary equilibrium. Our result also survives when agents have non-time-additive preferences if the consumption aggregator is a homogeneous function.

In addition, we show that adding markets does not always lead to more risk sharing. In particular, if the logarithm of aggregate consumption follows a random walk, allowing agents to trade claims on payoffs that are contingent on *aggregate shocks,* in addition to the risk-free bond, does not help them to smooth their consumption. Introducing these assets leaves interest rates and asset prices unaltered. However, if there is predictability in aggregate consumption growth, agents want to hedge their portfolio against interest rate shocks, creating a role for trade in a richer menu of assets. The risk premium irrelevance result, however, still applies. Finally, we also show that idiosyncratic uninsurable income

[1] Weil's (1989) statement of the risk-free rate puzzle, on the contrary, does not.

risk does not contribute any variation in the conditional market price of risk, beyond what is built into aggregate consumption growth.

In the quest towards the resolution of the equity premium puzzle identified by Hansen and Singleton (1983) and Mehra and Prescott (1985), uninsurable idiosyncratic income risk has been introduced into standard dynamic general equilibrium models.[2] Incomplete insurance of household consumption against idiosyncratic income risk was believed to increase the aggregate risk premium. Heaton and Lucas (1996), in their seminal paper, motivate their analysis as follows:

> The motivation for considering the interaction between trading frictions and asset prices in this environment is best understood by reviewing the findings of a number of recent papers. Telmer (1993) and Lucas (1994) examine a similar model with transitory idiosyncratic shocks and without trading costs. Surprisingly, they find that even though agents cannot insure against idiosyncratic shocks, predicted asset prices are similar to those with complete markets.' This occurs because when idiosyncratic shocks are transitory, consumption can be effectively smoothed by accumulating financial assets after good shocks and selling assets after bad shocks.

Heaton and Lucas (1996) attribute the failure of incomplete market models (or their partial success) in matching moments of asset prices to the household's success in smoothing consumption in this class of models. Lucas (1994) concludes that:

> agents effectively self-insure by trading to offset the idiosyncratic shocks when they have limited access to either a stock market or a bond market. Thus, even limited access to capital markets implies that asset prices will be similar to those predicted by the representative agent model, a finding that deepens the equity premium puzzle.

Our paper shows *analytically* that there is essentially no link between the extent to which idiosyncratic risk can be traded away by households and the size of the risk

[2]For examples, see the work of Ayiagari and Gertler (1991), Telmer (1993), Lucas (1994), Heaton and Lucas (1996), Krusell and Smith (1998), and Marcet and Singleton (1999).

premium. The main contribution of our paper is to argue that Telmer (1993), Lucas (1994), Heaton and Lucas (1996), Marcet and Singleton (1999) and others in this literature have reached the right conclusions -namely that adding uninsurable idiosyncratic income risk to standard models does not alter the asset pricing implications of the model - but not for the right reasons. These authors have argued that households manage to self-insure quite well by trading a single bond, and as result, the risk premium is not affected by aggregate risk.[3] We show analytically that, as long as the distribution of idiosyncratic shocks is independent of aggregate shocks, the extent of self-insurance does not matter. We can make our solvency constraints arbitrarily tight or make the income process highly persistent, and our theoretical result still goes through.

Recently Krusell and Smith (1997) and Storesletten, Telmer and Yaron (2006) have argued that models with idiosyncratic income shocks and incomplete markets can generate an equity premium that is substantially larger than the CCAPM if the variance of the idiosyncratic shock increases in recessions, a condition that Storesletten, Telmer and Yaron (2004) argue is present in the data. One way to interpret our findings is that we demonstrate analytically that such counter-cyclical cross-sectional variance of individual income shocks is not only sufficient, but *necessary* to make uninsurable idiosyncratic income shocks potentially useful for explaining the equity premium. Mankiw (1986) already showed in his analysis of the standard incomplete markets Euler equation that if the marginal utility of consumption in convex (a property that CRRA utility and also CARA utility satisfies) then *if* the cross-sectional variance of equilibrium household consumption growth varies over time, this is one recipe for increasing risk premia. Cross-sectional income variance that varies with the cycle is one way to induce time variation in the equilibrium consumption growth distribution across households. Our work shows that solvency constraints and transaction costs in incomplete market models alone, without cyclical cross-sectional income variance, cannot produce this time variation as an equilibrium outcome.

[3]In his survey of the equity premium literature, Kocherlakota (1996) concludes that:

> ..given this ability to self-insure, the behavior of the risk-free rate and the equity premium remain largely unaffected by the absence of markets as long as idiosyncratic shocks are not highly persistent.**?**

As a crucial step in our analysis of asset prices we derive a result that may be of independent theoretical interest. We show that allocations and prices in a class of models with idiosyncratic and aggregate risk as well as incomplete markets can be backed out from the allocations and interest rates in a stationary equilibrium of a model with only idiosyncratic risk (as in Bewley (1986), Huggett (1993) or Aiyagari (1994)). This result delivers an algorithm for computing equilibria in this model which appears to be simpler than the auctioneer algorithm devised by Lucas (1994) and its extension to economies with a continuum of agents. Under a transformed, aggregate-risk-neutral probability measure, there is a stationary recursive equilibrium for our economy characterized by an invariant measure over wealth and endowments, despite the presence of aggregate shocks. Hence, there is no need for computing a law of motion for this measure, or approximating it by a finite number of moments, as in Krusell and Smith (1997, 1998).

The paper is structured as follows. In section 2, we lay out the physical environment of our model. This section also demonstrates how to transform the stochastically growing economy into a stationary economy with a constant endowment. In section 3 we study this stationary model, called the Bewley model. The next section introduces the Arrow model, the model with aggregate uncertainty and a full set of Arrow securities whose payoffs are contingent on the realization of the aggregate shock. We show that a stationary equilibrium of the Bewley model can be mapped into an equilibrium of the Arrow economy just by scaling up allocations by the aggregate endowment. In section 5 we derive the same result for a model where only a one-period risk-free bond can be traded. We call this the *HL* model (for *h*ighly *l*imited asset model or *H*eaton-*L*ucas model). After briefly discussing the classic *L*ucas-*B*reeden representative agent model (henceforth *LB* model), section 6 shows that risk premia in the representative agent model and the Arrow model (and by implication, in the *HL* model) are identical. Section 7 investigates the robustness of our results with respect to the assumptions about the underlying stochastic income process, and shows in particular that most of our results can be extended to the case where the aggregate shocks is correlated over time and where preferences are not time-separable, but rather follow an Epstein-Zin specification. Finally, section 8 concludes; all proofs are contained in the appendix.

2 Environment

Our exchange economy is populated by a continuum of individuals of measure 1. There is a single nonstorable consumption good. The aggregate endowment of this good is stochastic. Each individual's endowment depends on the realization of an idiosyncratic and an aggregate shock. This economy is identical to the one described by Lucas (1994), except that ours is populated by a continuum of agents (as in Bewley (1986), Aiyagari and Gertler (1991), Huggett (1993) and Aiyagari (1994)), instead of just two agents.

2.1 Representation of Uncertainty

We denote the current aggregate shock by $z_t \in Z$ and the current idiosyncratic shock by $y_t \in Y$. For simplicity, both Z and Y are assumed to be finite. Furthermore, let $z^t = (z_0, \ldots, z_t)$ and $y^t = (y_0, \ldots, y_t)$ denote the history of aggregate and idiosyncratic shocks. As shorthand notation, we use $s_t = (y_t, z_t)$ and $s^t = (y^t, z^t)$. We let the economy start at initial node z_0. Conditional on idiosyncratic shock y_0 and thus $s_0 = (y_0, z_0)$, the probability of a history s^t is given by $\pi_t(s^t|s_0)$. We assume that these shocks follow a first order Markov process with transition probabilities given by $\pi(s'|s)$.

2.2 Preferences and Endowments

Consumers rank stochastic consumption streams $\{c_t(s^t)\}$ according to the following homothetic utility function:

$$U(c)(s_0) = \sum_{t=0}^{\infty} \sum_{s^t \geq s^0} \beta^t \pi(s^t|s_0) \frac{c_t(s^t)}{1-\gamma}^{1-\gamma} \tag{1}$$

where $\gamma > 0$ is the coefficient of relative risk aversion and $\beta \in (0, 1)$ is the constant time discount factor. We define $U(c)(s^t)$ to be the continuation expected lifetime utility from consumption allocation $c = \{c_t(s^t)\}$ in node s^t. This utility can be constructed recursively as follows:

$$U(c)(s^t) = u(c_t(s^t)) + \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) U(c)(s^t, s_{t+1})$$

where we made use of the Markov property the underlying stochastic processes.

The economy's aggregate endowment process $\{e_t\}$ depends only on the aggregate event history; we let $e_t(z^t)$ denote the aggregate endowment at node z^t. Each agent draws a 'labor income' share $\eta(y_t, z_t)$ as a fraction of the aggregate endowment in each period. Her labor income share only depends on the current individual and aggregate event. We denote the resulting individual labor income process by $\{\eta_t\}$, with

$$\eta_t(s^t) = \eta(y_t, z_t)e_t(z^t) \tag{2}$$

where $s^t = (s^{t-1}, y_t, z_t)$. We assume $\eta(y_t, z_t) > 0$ in all states of the world.

The stochastic growth rate of the endowment of the economy is denoted $\lambda(z^{t+1}) = e_{t+1}(z^{t+1})/e_t(z^t)$. We assume that aggregate endowment growth only depends on the current aggregate state:

Condition 2.1. *Aggregate endowment growth is a function of the current aggregate shock only:*

$$\lambda(z^{t+1}) = \lambda(z_{t+1})$$

Furthermore, we assume that a Law of Large Numbers holds[4], so that $\pi(s^t|s_0)$ is not only a household's individual probability of receiving income $\eta_t(s^t)$, but also the fraction of the population having that income. We do *not* assume that the idiosyncratic shocks y are uncorrelated over time.

In addition, there is a Lucas tree that yields a constant share α of the total aggregate endowment as capital income, so that the total dividends of the tree are given by $\alpha e_t(z^t)$ in each period. The remaining fraction of the total endowment accrues to individuals as labor income, so that $1 - \alpha$ denotes the labor income share. Therefore, by construction, the labor share of the aggregate endowment equals the sum over all individual labor income shares:

$$\sum_{y_t \in Y} \Pi_{z_t}(y_t)\eta(y_t, z_t) = (1 - \alpha), \tag{3}$$

[4]See e.g. Hammond and Sun (2003) for conditions under which a LLN holds with a continuum of random variables.

for all z_t, where $\Pi_{z_t}(y_t)$ represents the cross-sectional distribution of idiosyncratic shocks y_t, conditional on the aggregate shock z_t. By the law of large numbers, the fraction of agents who draw y in state z only depends on z. An increase in the capital income share α translates into proportionally lower individual labor income shares $\eta(y, z)$ for all (y, z).[5]

At time 0, the agents are endowed with initial wealth θ_0. This wealth represents the value of an agent's share of the Lucas tree producing the dividend flow in units of time 0 consumption, as well as the value of her labor endowment at date 0. We use Θ_0 to denote the initial joint distribution of wealth and idiosyncratic shocks (θ_0, y_0).

This concludes our description of the physical environment. Many of our results are derived using a de-trended version of our economy, with a constant aggregate endowment and a growth-adjusted transition probability matrix. The agents in this de-trended economy, discussed now, have stochastic time discount factors.

2.3 Transformation of Growth Economy into a Stationary Economy

We can transform our growing economy into a stationary economy with a stochastic time discount rate and a growth-adjusted probability matrix, following Alvarez and Jermann (2001), who use this transformation in a complete markets economy. First, we define growth deflated consumption allocations (or consumption shares) as

$$\hat{c}_t(s^t) = \frac{c_t(s^t)}{e_t(z^t)} \text{ for all } s^t. \tag{4}$$

Next, we define the *growth-adjusted* probabilities and the growth-adjusted discount factor as:

$$\hat{\pi}(s_{t+1}|s_t) = \frac{\pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\gamma}}{\sum_{s_{t+1}} \pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\gamma}} \text{ and } \hat{\beta}(s_t) = \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\gamma}.$$

[5] Our setup nests the baseline calibration of Heaton and Lucas (1996), except for the fact that they allow for variation in the capital share α depending on z.

Note that $\hat{\pi}$ is a well-defined Markov matrix in that $\sum_{s_{t+1}} \hat{\pi}(s_{t+1}|s_t) = 1$ for all s_t and that $\hat{\beta}(s_t)$ is stochastic as long as the original Markov process is not *iid* over time. Finally, let $\hat{U}(\hat{c})(s^t)$ denote the lifetime expected continuation utility in node s^t, under the new transition probabilities and discount factor, defined over consumption shares $\{\hat{c}_t(s^t)\}$:

$$\hat{U}(\hat{c})(s^t) = u(\hat{c}_t(s^t)) + \hat{\beta}(s_t) \sum_{s_{t+1}} \hat{\pi}(s_{t+1}|s_t)\hat{U}(\hat{c})(s^t, s_{t+1}) \tag{5}$$

In the appendix we prove that this transformation does not alter the agents' ranking of different consumption streams.

Proposition 2.1. *Households rank consumption share allocations in the de-trended economy in exactly the same way as they rank the corresponding consumption allocations in the original growing economy: for any s^t and any two consumption allocations c, c'*

$$U(c)(s^t) \geq U(c')(s^t) \Longleftrightarrow \hat{U}(\hat{c})(s^t) \geq \hat{U}(\hat{c}')(s^t)$$

where the transformation of consumption into consumption shares is given by (4).

This result is crucial for demonstrating that equilibrium allocations c for the stochastically growing economy can be found by solving for equilibrium allocations $\hat{c}$ in the transformed economy.

2.4 Independence of Idiosyncratic Shocks from Aggregate Conditions

Next, we assume that idiosyncratic shocks are independent of the aggregate shocks. This assumption is crucial for most of the results in this paper.

Condition 2.2. *Individual endowment shares $\eta(y_t, z_t)$ are functions of the current idiosyncratic state y_t only, that is $\eta(y_t, z_t) = \eta(y_t)$. Also, transition probabilities of the shocks can be decomposed as*

$$\pi(z_{t+1}, y_{t+1}|z_t, y_t) = \varphi(y_{t+1}|y_t)\phi(z_{t+1}|z_t).$$

That is, individual endowment *shares* and the transition probabilities of the idiosyncratic shocks are independent of the aggregate state of the economy z. In this case, the growth-adjusted probability matrix $\hat{\pi}$ and the re-scaled discount factor is obtained by adjusting only the transition probabilities for the *aggregate* shock, ϕ, but not the transition probabilities for the idiosyncratic shocks:

$$\hat{\pi}(s_{t+1}|s_t) = \varphi(y_{t+1}|y_t)\hat{\phi}(z_{t+1}|z_t), \text{ and } \hat{\phi}(z_{t+1}|z_t) = \frac{\phi(z_{t+1}|z_t)\lambda(z_{t+1})^{1-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{1-\gamma}}.$$

Furthermore, the growth-adjusted discount factor only depends on the aggregate state z_t:

$$\hat{\beta}(z_t) = \beta \sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{1-\gamma} \tag{6}$$

The first part of our analysis assumes that the aggregate shocks are independent over time:

Condition 2.3. *Aggregate endowment growth is i.i.d.:*

$$\phi(z_{t+1}|z_t) = \phi(z_{t+1}).$$

In this case the growth rate of aggregate endowment is uncorrelated over time, so that the logarithm of the aggregate endowment follows a random walk with drift.[6] As a result, the growth-adjusted discount factor is a constant: $\hat{\beta}(z_t) = \hat{\beta}$, since $\hat{\phi}(z_{t+1}|z_t) = \hat{\phi}(z_{t+1})$.

There are two competing effects on the growth-adjusted discount rate: consumption growth itself makes agents more impatient, while the consumption risk makes them more patient.[7]

[6] In section 7 we show that most of our results survive the introduction of persistence in the growth rates if a complete set of contingent claims on aggregate shocks is traded.

[7] This growth-adjusted measure is obviously connected to the risk-neutral measure commonly used in asset pricing (see e.g. Harrison and Kreps, 1979). Under our hatted measure, agents can evaluate utils from consumption streams while abstracting from aggregate risk; under a risk-neutral measure, agents can price payoffs by simply discounting at the risk-free rate.

2.5 A Quartet of Economies

In order to derive our results, we study four models, whose main characteristics are summarized in table 1. The first three models are endowment economies with aggregate shocks. The models differ along two dimensions, namely whether agents can trade a full set of Arrow securities against aggregate shocks, and whether agents face idiosyncratic risk, in addition to aggregate risk. Idiosyncratic risk, if there is any, is never directly insurable.

Table 1: Summary of Four Economies

Model	*Aggregate Shocks*	*Idiosyncr. Shocks*	*Arrow Securities*
HL	Yes	Yes	No
Arrow	Yes	Yes	Yes
BL	Yes	No	Yes
Bewley	No	Yes	N/A

Our primary goal is to understand asset prices in the first model in the table, the *HL* model. This model has idiosyncratic and aggregate risk, as well as incomplete markets. We call this the *HL* model since besides using stocks agents can only insure against idiosyncratic and aggregate shocks by trading a single bond.

The standard representative agent complete markets Breeden (1979)-Lucas (1978) (*BL*) model lies on the other end of the spectrum, in that there is no idiosyncratic risk and a full menu of Arrow securities for the representative agent to hedge against aggregate risk. Through our analysis we will demonstrate that in the *HL* model the standard representative agent Euler equation for excess returns is satisfied:

$$E_t\left[\beta\left(\lambda_{t+1}\right)^{-\gamma}\left(R^s_{t+1}-R_t\right)\right]=0 \tag{7}$$

where R^s_{t+1} is the return on the stock, R_t is the return on the bond and λ_{t+1} is the growth rate of the aggregate endowment. Hence, the aggregate risk premium is identical in the *HL* and the *BL* model. Constantinides (1982) had already shown that, in the case of complete markets, even if agents are heterogeneous in wealth, there exists a representative agent who satisfies the Euler equation for excess returns (7) and also the Euler equation for bonds:

$$E_t \left[\beta \left(\lambda_{t+1}\right)^{-\gamma} R_t\right] = 1. \tag{8}$$

The key to Constantinides' result is that markets are *complete*. We show that the first Euler equation in (7) survives market incompleteness and binding solvency constraints. The second one does not. To show this result, we employ a third model, the Arrow model (second row in the table). Here, households trade a full set of Arrow securities against aggregate risk, but not against idiosyncratic risk. The fundamental result underlying our asset pricing findings is that equilibria in both the *HL* and the Arrow model can be found by first determining equilibria in a model with *only* idiosyncratic risk (the Bewley model, fourth row in the table) and then by simply scaling consumption allocations in that model by the stochastic aggregate endowment.

Outline We therefore start in section 3 by characterizing equilibria for the Bewley model, a stationary economy with a constant aggregate endowment in which agents trade a single discount bond and a stock.[8] This model merely serves as a computational device. Then we turn to the stochastically growing economy (with different market structures), the one whose asset pricing implications we are interested in, and we show that equilibrium consumption allocations from the Bewley model can be implemented as equilibrium allocations in the stochastically growing *HL* and Arrow model.

3 The Bewley Model

In this economy the aggregate endowment is constant. The households face idiosyncratic shocks y that follow a Markov process with transition probabilities $\varphi(y'|y)$. The household's preferences over consumption shares $\{\hat{c}(y^t)\}$ are as defined in equation (5), with the time discount factor $\hat{\beta}$ as defined in equation (6). The adjusted discount factor is $\hat{\beta}$

[8] One of the two assets will be redundant for the households, so that this model is a standard Bewley model, as studied by Bewley (1986), Huggett (1993) or Aiyagari (1994). The presence of both assets will make it easier to demonstrate our equivalence results with respect to the Bond and Arrow model later on.

constant, because the aggregate shocks are i.i.d. (see Condition (2.3)).

3.1 Market Structure

Agents trade only a riskless discount bond and shares in a Lucas tree that yields safe dividends of α in every period. The price of the Lucas tree at time t is denoted by v_t.[9] The riskless bond is in zero net supply. Each household is indexed by an initial condition (θ_0, y_0), where θ_0 denotes its wealth (including period 0 labor income) at time 0.

The household chooses consumption $\{\hat{c}_t(\theta_0, y^t)\}$, bond positions $\{\hat{a}_t(\theta_0, y^t)\}$ and share holdings $\{\hat{\sigma}_t(\theta_0, y^t)\}$ to maximize its normalized expected utility $\hat{U}(\hat{c})(s^0)$, subject to a standard budget constraint:[10]

$$\hat{c}_t(y^t) + \frac{\hat{a}_t(y^t)}{\hat{R}_t} + \hat{\sigma}_t(y^t)\hat{v}_t = \eta(y_t) + \hat{a}_{t-1}(y^{t-1}) + \hat{\sigma}_{t-1}(y^{t-1})(\hat{v}_t + \alpha)$$

Finally, each household faces one of two types of borrowing constraints. The first one restricts household wealth at the end of the current period. The second one restricts household wealth at the beginning of the next period:[11]

$$\begin{aligned} \frac{\hat{a}_t(y^t)}{\hat{R}_t} + \hat{\sigma}_t(y^t)\hat{v}_t &\geq \hat{K}_t(y^t) \text{ for all } y^t \\ \hat{a}_t(y^t) + \hat{\sigma}_t(y^t)(\hat{v}_{t+1} + \alpha) &\geq \hat{M}_t(y^t) \text{ for all } y^t \end{aligned}$$

3.2 Equilibrium in the Bewley Model

The definition of equilibrium in this model is standard.

Definition 3.1. *For an initial distribution Θ_0 over (θ_0, y_0), a competitive equilibrium for the Bewley model consists of trading strategies $\{\hat{a}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$, consumption allocations $\{\hat{c}_t(\theta_0, y^t)\}$, interest rates and share prices $\{\hat{R}_t, \hat{v}_t\}$ such that*

1. *Given prices, allocations solve the household maximization problem*

[9] The price of the tree is nonstochastic due to the absence of aggregate risk.

[10] We suppress dependence on θ_0 for simplicity whenever there is no room for confusion.

[11] This distinction is redundant in the Bewley model, but it will become meaningful in our models with stochastically growing endowment.

2. The goods markets and asset markets clear in all periods t

$$
\begin{aligned}
\int \sum_{y^t} \varphi(y^t|y_0)\hat{c}_t(\theta_0, y^t)d\Theta_0 &= 1 \\
\int \sum_{y^t} \varphi(y^t|y_0)\hat{a}_t(\theta_0, y^t)d\Theta_0 &= 0 \\
\int \sum_{y^t} \varphi(y^t|y_0)\hat{\sigma}_t(\theta_0, y^t)d\Theta_0 &= 1
\end{aligned}
$$

Without aggregate risk, the risk-free one period bond and the risk-free stock are perfect substitutes for households, so that the absence of arbitrage implies the following restriction on equilibrium stock prices and interest rates:

$$\hat{R}_t = \frac{\hat{v}_{t+1} + \alpha}{\hat{v}_t}$$

In addition, household portfolios are indeterminate, and without loss of generality we focus on the case in which households simply trade the stock, not the bond: $\hat{a}_t(\theta_0, y^t) \equiv 0$.[12]

A stationary Bewley equilibrium is a constant interest rate $\hat{R}$, a share price $\hat{v}$, optimal household allocations and a time-invariant measure Φ over income shocks and financial wealth. We define a stationary recursive competitive equilibrium in section (A.1) of the appendix, and also outline a (straightforward) algorithm for computing it there.

4 Arrow Economy

We now turn to our main object of interest, the stochastically growing economy. We start off with the Arrow market structure in which households can trade the stock and a complete menu of contingent claims on aggregate shocks. Idiosyncratic shocks are still uninsurable. We demonstrate in this section that the allocations we compute for the stationary Bewley economy can be mapped into equilibrium allocations and prices in the

[12] Alternatively, we could have agents simply trade in the bond and adjust the net supply of bonds to account for the positive capital income α in the aggregate. We only introduce both assets into the Bewley economy to make the mapping to allocations in the Arrow and HL models simpler.

stochastically growing Arrow economy.

4.1 Trading

Let $a_t(s^t, z_{t+1})$ denote the quantity purchased of a security that pays off one unit of the consumption good if aggregate shock in the next period is z_{t+1}, irrespective of the idiosyncratic shock y_{t+1}. Its price today is given by $q_t(z^t, z_{t+1})$. In addition, households trade shares in the Lucas tree. We use $\sigma_t(s^t)$ to denote the number of shares a household with history $s^t = (y^t, z^t)$ purchases today and we let $v_t(z^t)$ denote the price of one share.

An agent starting period t with initial wealth $\theta_t(s^t)$ buys consumption commodities in the spot market and trades securities subject to the usual budget constraint:

$$c_t(s^t) + \sum_{z_{t+1}} a_t(s^t, z_{t+1}) q_t(z^t, z_{t+1}) + \sigma_t(s^t) v_t(z^t) \leq \theta_t(s^t). \tag{9}$$

In addition to the budget constraints, the households' trading strategies are subject to solvency constraints of one of two types. The first type of constraint imposes a lower bound on the value of the asset portfolio at the end of the period today,

$$\sum_{z_{t+1}} a_t(s^t, z_{t+1}) q_t(z^t, z_{t+1}) + \sigma_t(s^t) v_t(z^t) \geq K_t(s^t), \tag{10}$$

while the second type imposes state-by-state lower bounds on net wealth tomorrow,

$$a_t(s^t, z_{t+1}) + \sigma_t(s^t) \left[v_{t+1}(z^{t+1}) + \alpha e_{t+1}(z_{t+1}) \right] \geq M_t(s^t, z_{t+1}) \text{ for all } z_{t+1}. \tag{11}$$

We assume these solvency constraints are at least as tight as the natural borrowing constraints. This is sufficient to prevent Ponzi schemes. In addition, we impose restrictions on the solvency constraints that make them proportional to the aggregate endowment in the economy:

Condition 4.1. *We assume the borrowing constraints only depend on the aggregate his-*

tory through the level of the aggregate endowment. That is, we assume

$$K_t(y^t, z^t) = \hat{K}_t(y^t)e_t(z^t)$$

and

$$M_t(y^t, z^t, z_{t+1}) = \hat{M}_t(y^t)e_{t+1}(z^{t+1}).$$

If the constraints did not have this feature in a stochastically growing economy, the constraints would become more or less binding as the economy grows, clearly not a desirable feature[13]. The definition of an equilibrium is completely standard (see section A.2 of the Appendix). Instead of working in the growing economy, we immediately transform the Arrow economy into a stationary economy. We use hatted variables to denote the variables in the stationary economy. In the next subsection, we deflate allocations by the aggregate endowment and we demonstrate that the resulting equilibrium allocations are the same as the allocations in a Bewley equilibrium.

4.2 Equilibrium in the De-trended Arrow Model

Households rank consumption shares $\{\hat{c}_t\}$ in exactly the same way as original consumption streams $\{c_t\}$, so we can solve for an equilibrium in the de-trended economy. Dividing the budget constraint (27) by $e_t(z^t)$, and using equation (28) yields the deflated budget constraint:

$$\begin{aligned} & \hat{c}_t(s^t) + \sum_{z_{t+1}} \hat{a}_t(s^t, z_{t+1})\hat{q}_t(z^t, z_{t+1}) + \sigma_t(s^t)\hat{v}_t(z^t) \\ \leq \quad & \eta(y_t) + \hat{a}_{t-1}(s^{t-1}, z_t) + \sigma_{t-1}(s^{t-1})\left[\hat{v}_t(z^t) + \alpha\right], \end{aligned} \tag{12}$$

[13]It is easy to show that solvency constraints that are not too tight in the sense of Alvarez and Jermann (2000) satisfy this condition. In the incomplete markets literature, the borrowing constraints usually have this feature (see e.g. Heaton and Lucas (1996)). If not, one cannot compute a stationary equilibrium. It is more common in this literature to impose growing short-sale constraints on stocks and bonds separately instead of total financial wealth, but this is done mostly for computational reasons, to bound the state space. In fact, the bounds should be directly on total financial wealth instead, if the solvency constraints are to prevent default (see e.g. Zhang (1996) and Alvarez and Jermann (2000)).

where we have defined the deflated Arrow positions $\hat{a}_t(s^t, z_{t+1}) = \frac{a_t(s^t, z_{t+1})}{e_{t+1}(z^{t+1})}$ and prices $\hat{q}_t(z^t, z_{t+1}) = q_t(z^t, z_{t+1})\lambda(z_{t+1})$. The deflated stock prices are given by $\hat{v}_t(z^t) = \frac{v_t(z^t)}{e_t(z^t)}$. Similarly, by deflating the solvency constraints (10) and (11) using condition (4.1) yields:

$$\sum_{z_{t+1}} \hat{a}_t(s^t, z_{t+1})\hat{q}_t(z^t, z_{t+1}) + \sigma_t(s^t)\hat{v}_t(z^t) \geq \hat{K}_t(y^t) \tag{13}$$

$$\hat{a}_t(s^t, z_{t+1}) + \sigma_t(s^t)\left[\hat{v}_{t+1}(z^{t+1}) + \alpha\right] \geq \hat{M}_t(y^t) \text{ for all } z_{t+1}. \tag{14}$$

Finally, the goods market clearing condition is given by[14]:

$$\int \sum_{y^t} \pi(y^t|y_0)\hat{c}_t(\theta_0, s^t)d\Theta_0 = 1. \tag{15}$$

The asset market clearing conditions are exactly the same as before. In the stationary economy, the household maximizes $\hat{U}(\hat{c})(s_0)$ by choosing consumption, Arrow securities and shares of the Lucas tree, subject to the budget constraint (12) and the solvency constraint (13) or (14) in each node s^t. The definition of a competitive equilibrium in the de-trended Arrow economy is straightforward.

Definition 4.1. *For initial aggregate state z_0 and distribution Θ_0 over (θ_0, y_0), a competitive equilibrium for the de-trended Arrow economy consists of trading strategies $\{\hat{a}_t(\theta_0, s^t, z_{t+1})\}$, $\{\hat{\sigma}_t(\theta_0, s^t)\}$, $\{\hat{c}_t(\theta_0, s^t)\}$ and prices $\{\hat{q}_t(z^t, z_{t+1})\}$, $\{\hat{v}_t(z^t)\}$ such that*

1. *Given prices, allocations solve the household maximization problem*
2. *The goods market clears, that is, equation* (15) *holds for all z^t.*
3. *The asset markets clear*

$$\begin{aligned}
\int \sum_{y^t} \varphi(y^t|y_0)\hat{\sigma}_t(\theta_0, s^t)d\Theta_0 &= 1 \\
\int \sum_{y^t} \varphi(y^t|y_0)\hat{a}_t(\theta_0, s^t, z_{t+1})d\Theta_0 &= 0 \textit{ for all } z_{t+1} \in Z
\end{aligned}$$

[14]The conditional probabilities simplify due to condition (2.2).

The first order conditions and complementary slackness conditions, together with the appropriate transversality condition, are listed in the appendix in section (A.3). These are necessary and sufficient conditions for optimality on the household side. Now, we are ready to establish the equivalence between equilibria in the Bewley model and in the Arrow model.

4.3 Equivalence Results

Our first result states that if the consumption shares in the de-trended economy do not depend on the aggregate history z^t, then it follows that the interest rates in this economy are deterministic.

Proposition 4.1. *In the de-trended Arrow economy, if there exists a competitive equilibrium with equilibrium consumption allocations* $\{\hat{c}_t(\theta_0, y^t)\}$ *, then there is a deterministic interest rate process* $\left\{\hat{R}_t^A\right\}$ *and equilibrium prices* $\{\hat{q}_t(z^t, z_{t+1})\}$ *, that satisfy:*

$$\hat{q}_t(z^t, z_{t+1}) = \frac{\hat{\phi}(z_{t+1})}{\hat{R}_t^A} \tag{16}$$

The interest rate in the deflated economy can depend on time, but not on the aggregate state of the economy, simply because nothing else in the Euler equation does:

$$1 \quad = \quad \hat{\beta}\hat{R}_t^A \underbrace{\sum_{z_{t+1}} \hat{\phi}(z_{t+1})}_{drops\ out} \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \tag{17, 18}$$

The aggregate shocks z do not affect the transformed discount factor $\hat{\beta}$, because of the i.i.d assumption, and the transition probabilities for the aggregate shocks disappear from the Euler equation altogether, because they are independent from the y shocks.[15] By summing over aggregate states tomorrow on both sides of equation (16), we can compute

[15]Finally, the dependence of $\hat{R}_t^A$ on time t is not surprising since, for an arbitrary initial distribution of assets Θ_0, we cannot expect the equilibrium to be stationary. In the same way we expect that $\hat{v}_t(z^t)$ is only a function of t as well, but not of z^t.

the interest rate:

$$\hat{R}_t^A = \frac{1}{\sum_{z_{t+1}} \hat{q}_t(z^t, z_{t+1})}.$$

This interest rate in the Arrow model is also the equilibrium interest rate in the Bewley model of section (3). Once we have found interest rates for the de-trended economy, $\hat{R}_t^A$, we can back out the implied interest rate for the original growing Arrow economy.

Corollary 4.1. *Risk-free interest rates in the original Arrow model are given by*

$$R_t^A = \hat{R}_t^A * \frac{\sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{1-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{-\gamma}} \tag{19}$$

In the absence of aggregate risk (constant λ), the risk-free rates in the original and deflated economy are related by $R_t^A = \hat{R}_t^A \lambda_{t+1}$where λ_{t+1} is the gross growth rate of endowment between period t and $t+1$.

Trading Strategies The next question is which trading strategies for Arrow securities would support consumption allocations in the de-trended Arrow model that do not depend on aggregate shocks as well, that is $\hat{c}_t(\theta_0, s^t) = \hat{c}_t(\theta_0, y^t)$. The next corollary provides the answer.

Corollary 4.2. *This equilibrium is supported by aggregate history-invariant Arrow security trades, that is $\hat{a}_t(\theta_0, s^t, z_{t+1}) = \hat{a}_t(\theta_0, y^t)$ for all z^{t+1}.*

This is not surprising, because the consumption shares do not depend on the aggregate history. This brings us to the main result, the mapping between the stochastically growing Arrow economy and the Bewley economy.

Theorem 4.1. *An equilibrium of the Bewley model $\{\hat{c}_t(\theta_0, y^t), \hat{a}_t(\theta_0, y^t),\ \hat{\sigma}_t(\theta_0, y^t)\}$ and $\{\hat{R}_t, \hat{v}_t\}$ can be made into an equilibrium for the Arrow economy with growth, $\{a_t(\theta_0, s^t, z_{t+1})\}$,*

$\{\sigma_t(\theta_0, s^t)\}$, $\{c_t(\theta_0, s^t)\}$ *and* $\{q_t(z^t, z_{t+1})\}$, $\{v_t(z^t)\}$, *with*

$$\begin{aligned}
c_t(\theta_0, s^t) &= \hat{c}_t(\theta_0, y^t) e_t(z^t) \\
\sigma_t(\theta_0, s^t) &= \hat{\sigma}_t(\theta_0, y^t) \\
a_t(\theta_0, s^t, z_{t+1}) &= \hat{a}_t(\theta_0, y^t) e_{t+1}(z^{t+1}) \\
v_t(z^t) &= \hat{v}_t e_t(z^t) \\
q_t(z^t, z_{t+1}) &= \frac{1}{\hat{R}_t} * \frac{\phi(z_{t+1})\lambda(z_{t+1})^{-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1})\lambda(z_{t+1})^{1-\gamma}}
\end{aligned}$$

We can solve for an equilibrium in the Bewley economy (section 3), including the risk free interest rate $\hat{R}_t$, and we can deduce the equilibrium allocations and prices for the Arrow economy from those in the Bewley economy. The key to this result is that households in the Bewley model face exactly the same Euler equations as the households in the de-trended version of the Arrow economy (as in equation (17)).

This result has several important implications. First, the existence proofs in the literature for equilibria in Bewley economies directly carry over to the stochastically growing economy[16]. Second, the moments of the wealth distribution vary over time but proportionally to the aggregate endowment: e.g. the ratio of the mean to the standard deviation of the wealth distribution is constant if we start the corresponding Bewley economy off from its invariant wealth distribution. Third, without loss of generality, we can focus on equilibria in the Arrow model in which Arrow securities are not traded.

Corollary 4.3. *We can support an equilibrium in the Arrow economy without any trade in the contingent claims market:*

$$\hat{a}_t(\theta_0, s^t, z_{t+1}) = \hat{a}_t(\theta_0, y^t) = 0 \textit{ for all } z^{t+1}.$$

The proof is obvious. In the Bewley model, the bonds are a redundant asset. Therefore it follows from our equivalence result that we can support the equilibrium allocation in the

[16]To prove existence, Aiyagari (1994) assumes i.i.d. idiosyncratic shocks but Huggett (1993) does not. Under regularity conditions, Bewley equilibria are also unique. This implies there is a unique Arrow equilibrium that corresponds to a Bewley equilibrium in the deflated version of the economy, but we cannot rule out other Arrow equilibria in general.

Arrow model with the following Arrow security trades $\hat{a}_t(\theta_0, s^t, z_{t+1}) = \hat{a}_t(\theta_0, y^t) = 0$ for all z^{t+1}.This no-trade corollary suggests that the equivalence will carry over to economies with more limited asset structures. That is what we show in the next section.

We will use this equivalence result with the Bewley model to show that asset prices in the Arrow economy are identical to those in the representative agent economy, except for the lower interest rate (and a higher price/dividend ratio for stocks). Finally, we conclude this section by showing that the previous equivalence result does not depend on the absence of solvency constraints.

Corollary 4.4. *If $\hat{a}_t(\theta_0, y^t)$ and $\hat{\sigma}_t(\theta_0, y^t)$ satisfy the constraints in the de-trended Arrow model (equivalently, in the Bewley model), then $\sigma_t(\theta_0, s^t) = \hat{\sigma}_t(\theta_0, y^t)$ and $a_t(\theta_0, s^t, z_{t+1}) = \hat{a}_t(\theta_0, y^t) e_{t+1}(z^{t+1})$ also satisfy the solvency constraints in the stochastically growing Arrow model.*

Importantly, this is the case regardless of the tightness of the solvency constraints. Even if the borrowing constraints bind, nothing in the Euler equation of the de-trended Arrow economy depends on z:

$$\begin{aligned} 1 &= \hat{\beta} \hat{R}_t^A \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \\ &\quad + \mu_t(y^t) + \hat{R}_t^A \kappa_t(y^t) \ \forall z_{t+1}, \end{aligned}$$

where $\mu_t(y^t)$ and $\kappa_t(y^t)$ are the multipliers on the solvency constraints. These multipliers are obtained from the computation of the Bewley equilibrium, and hence do not depend on the aggregate history either.

5 *HL* Economy

We now turn our attention to the model whose asset pricing implications we are really interested in, namely the model with a stock and a single uncontingent bond. This section establishes the equivalence of equilibria in the *HL* model and the Bewley model by showing that optimality conditions in the de-trended Arrow and *HL* economy are identical. In addition, we show that agents do not even trade bonds in the benchmark case with i.i.d. aggregate endowment growth shocks.

5.1 Market Structure

In the *HL* economy, agents only trade a one-period discount bond and a stock. An agent who starts period t with initial wealth $\theta_t(s^t)$ buys consumption commodities in the spot market and trades a one-period bond and the stock, subject to budget constraint:

$$c_t(s^t) + \frac{b_t(s^t)}{R_t(z^t)} + \sigma_t(s^t)v_t(z^t) \leq \theta_t(s^t). \tag{20}$$

$b_t(s^t)$ denotes the amount of bonds purchased and $R_t(z^t)$ is the gross interest rate from period t to $t+1$. As was the case in the Arrow model, short-sales of the bond and the stock are constrained by a lower bound on the value of the portfolio today,

$$\frac{b_t(s^t)}{R_t(z^t)} + \sigma_t(s^t)v_t(z^t) \geq K_t(s^t), \tag{21}$$

or a state-by-state constraint on the value of the portfolio tomorrow,

$$b_t(s^t) + \sigma_t(s^t)\left[v_{t+1}(z^{t+1}) + \alpha e_{t+1}(z_{t+1})\right] \geq M_t(s^t, z_{t+1}) \text{ for all } z_{t+1}. \tag{22}$$

Since $b_t(s^t)$ and $\sigma_t(s^t)$ are chosen before z_{t+1} is realized, at most one of the constraints (22) will be binding at a given time. The definition of an equilibrium for the *HL* model follows directly. (see section (A.4) in the appendix).

We now show that the equilibria in the Arrow and the *HL* model coincide. As a corollary, it follows that the asset pricing implications of both models are identical. In order to do so we first transform the growing economy into a stationary, de-trended economy.

5.2 Equilibrium in the De-trended *HL* Model

Dividing the budget constraint (20) by $e_t(z^t)$ we obtain, using (32),

$$\hat{c}_t(s^t) + \frac{\hat{b}_t(s^t)}{R_t(z^t)} + \sigma_t(s^t)\hat{v}_t(z^t) \le \eta(y_t) + \frac{\hat{b}_{t-1}(s^{t-1})}{\lambda(z_t)} + \sigma_{t-1}(s^{t-1})\left[\hat{v}_t(z^t) + \alpha\right],$$

where we define the deflated bond position $\hat{b}_t(s^t) = \frac{b_t(s^t)}{e_t(z^t)}$. Using condition (4.1), the solvency constraints in the de-trended economy are simply:

$$\begin{aligned}\frac{\hat{b}_t(s^t)}{R_t(z^t)} + \sigma_t(s^t)\hat{v}_t(z^t) &\geq \hat{K}_t(y^t), \text{ or} \\ \frac{\hat{b}_t(s^t)}{\lambda(z_{t+1})} + \sigma_t(s^t)\left[\hat{v}_{t+1}(z^{t+1}) + \alpha\right] &\geq \hat{M}_t(y^t) \text{ for all } z_{t+1}.\end{aligned}$$

The definition of equilibrium in the de-trended *HL* model is straightforward and hence omitted[17]. We now show that equilibrium consumption allocations in the de-trended *HL* model coincide with those of the Arrow model. For simplicity we first abstract from binding borrowing constraints and then extend our results to that case later on.

5.3 Equivalence Results

The Euler equation for bonds in the de-trended *HL* model for an unconstrained household is given by:

$$1 = \hat{\beta}\hat{R}_t^{HL} \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))},$$

It is identical to the de-trended Arrow economy' Euler equation for bonds, as long as the equilibrium risk-free rate $\hat{R}_t^{HL}$ in the bond economy is the same as that in the de-trended Arrow economy. This explains the following proposition.

Proposition 5.1. *If borrowing constraints in both models never bind, the Euler equations in the de-trended* HL *model are identical to the Euler equations in the de-trended Arrow model. The equilibrium risk-free rates in both models coincide* $\left\{R_t^{HL}(z^t) = R_t^A(z^t)\right\}$.

[17] We list the first order conditions for household optimality and the transversality conditions in section (A.4) of the appendix.

Trading Strategies We now investigate the trading strategies that support the equilibrium consumption allocation in the *HL* model. Net wealth in the de-trended version of the *HL* model can only depend on the idiosyncratic shock history, as does de-trended consumption $\hat{c}_t(y^t)$

Lemma 5.1. *In the de-trended bond economy, net wealth only depends on* y^t.

This implies that in the growth economy with aggregate uncertainty, wealth at the beginning of the period,

$$b_{t-1}(s^{t-1}) + \sigma_{t-1}(s^{t-1}) \left[v_t(z^t) + \alpha e_t(z^t)\right] \tag{23}$$

has to be proportional to $e_t(z^t)$. From inspecting (23) it is obvious that this is only possible if $v_t(z^t)$ is proportional to aggregate endowment $e_t(z^t)$ and if $b_{t-1}(s^{t-1}) = 0$! In the appendix we therefore show the following proposition.

Proposition 5.2. *In the* HL *model, the bond market is inoperative:* $b_{t-1}(s^{t-1}) = \hat{b}_{t-1}(s^{t-1}) = 0$ *for all* s^{t-1}.

All of the consumption smoothing is done by trading stocks, because agents want to keep their net wealth proportional to the level of the aggregate endowment. The stock position of households in the *HL* model is simply given by the net wealth position in the Arrow economy $\hat{\theta}_t(y^{t-1})$ deflated by the cum dividend price-dividend ratio,

$$\sigma_{t-1}^{IC}(y^{t-1}) = \frac{\hat{\theta}_{t-1}(y^{t-1})}{[\hat{v}_t + \alpha]} = \frac{\hat{a}_{t-1}(y^{t-1})}{[\hat{v}_t + \alpha]} + \sigma_{t-1}^{A}(y^{t-1}),$$

where $\hat{\theta}_{t-1}(y^{t-1})$, as before, denotes the total net wealth position in the deflated Arrow economy at the start of t:[18]

$$\hat{a}_{t-1}(y^{t-1}) + \sigma_{t-1}^{A}(y^{t-1}) \left[\hat{v}_t + \alpha\right] = \hat{\theta}_{t-1}(y^{t-1}).$$

Also, it is important to note that the stock positions in the *HL* model are not necessarily equal to the stock positions in the Arrow economy. A household that is long in bonds

[18] Remember that in the deflated Arrow model households find it optimal to only trade state-uncontingent assets, see Corollary 4.2

in the Arrow economy simply has a higher stock position in the *HL* economy, while a household that is short in the bond market in the Arrow economy holds less stocks in the *HL* economy. The net borrowing and lending in the *HL* model is done in the stock market, not in the bond market[19]. Consequently, we have the following result:[20].

Theorem 5.1. *An equilibrium of a stationary Bewley economy* $\{\hat{c}_t(\theta_0, y^t), \hat{a}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$ *and* $\{\hat{R}_t, \hat{v}_t\}$ *can be made into an equilibrium for the* HL *economy with growth,* $\{b_t(\theta_0, s^t)\}$, $\{c_t(\theta_0, s^t)\}$, $\{\sigma_t(\theta_0, s^t)\}$ *and* $\{R_t(z^t)\}$ *and* $\{v_t(z^t)\}$ *where*

$$\begin{aligned} c_t(\theta_0, s^t) &= \hat{c}_t(\theta_0, y^t) e_t(z^t) \\ \sigma_t^{IC}(\theta_0, s^t) &= \hat{\sigma}_t^{IC}(\theta_0, y^t) = \frac{\hat{a}_t(\theta_0, y^t)}{[\hat{v}_{t+1} + \alpha]} + \sigma_t^A(\theta_0, y^t) \\ v_t(z^t) &= \hat{v}_t e_t(z^t) \\ R_t(z^t) &= \hat{R}_t(z^t) * \frac{\sum_{z_{t+1}} \phi(z_{t+1}|z_t) \lambda(z_{t+1})^{1-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1}|z_t) \lambda(z_{t+1})^{-\gamma}} \end{aligned}$$

and bond holdings given by $b_t(\theta_0, s^t) = 0$.

Absent any binding borrowing constraints, we can solve for equilibria in a standard Bewley model economy and then map this equilibrium into one for both the Arrow economy and the *HL* economy *with* aggregate uncertainty. The risk-free interest rate and the price of the Lucas tree coincide in the stochastic Arrow and *HL* economies. Finally, without loss of generality, we can restrict attention to equilibria in which bonds are not traded; consequently transaction costs in the bond market would not change our results. Transaction costs in the stock market of course would (see section (7)).

In the *HL* economy, households do not have a motive for trading bonds, unless there are short-sale constraints on stocks. We do not deal with this case. In addition, the no-trade result depends critically on the i.i.d assumption for aggregate shocks, as we will show in section (7). If the aggregate shocks are not i.i.d, agents want to hedge against the interest rate shocks -these show up as aggregate taste shocks in the de-trended economy.

[19] This explains why net wealth at the start of the period is proportional to $e_t(z^t)$: all the households are long or short in the stock.

[20] Even in the Arrow model, trade in bonds is not needed to implement the equilibrium consumption allocation, and thus we could consider only equilibria in the Arrow model with $\hat{a}_t(y^t) = 0$.

Borrowing constraints The next corollary establishes that our previous results survive the introduction of borrowing constraints.

Corollary 5.1. *The asset allocation for the HL model from the previous theorem satisfies the borrowing constraints* (21) *or* (22) *if and only if the associated asset allocation* $a_t(\theta_0, s^t, z_{t+1})$ *and* $\sigma_t^A(\theta_0, s^t)$ *satisfies the borrowing constraints in the stochastically growing Arrow economy*

This result follows from the fact that the new wealth at the end of the period and the beginning of the next period coincides in equilibria of both models. As a consequence, we have established that even with borrowing constraints, the equilibria in the Arrow and *HL* model coincide. Since we know that households only trade the stocks in equilibrium, our result is robust to the introduction of short-sale constraints on stock and bond holdings separately, as long as the short-sale constraints are not tighter than the solvency constraints $K(\cdot)$ or $M(\cdot)$.

We now compare these asset prices to those emerging from the *BL* (standard representative agent) model.

6 Asset Pricing Implications

This section shows that the multiplicative risk premium on a claim to aggregate consumption in the *HL* economy -and the Arrow economy- equals the risk premium in the representative agent economy. Idiosyncratic income risk only lowers the risk-free rate.

6.1 Consumption-CAPM

Our benchmark is the *BL* model. The representative agent owns a claim to the aggregate 'labor' income stream $\{(1-\alpha)e_t(z^t)\}$ and she can trade a stock (a claim to the dividends $\alpha e_t(z^t)$ of the Lucas tree), a bond and a complete set of Arrow securities[21].

First, we show that the Breeden-Lucas Consumption-CAPM also prices excess returns

[21] see section (A.5) in the Appendix for a complete description.

on the stock in the *HL* model and the Arrow model. R^s denotes the return on a claim to aggregate consumption.

Lemma 6.1. *The Consumption-CAPM prices excess returns in the Arrow economy and the* HL *Economy:*

$$E_t \left[\left(R^s_{t+1} - R_t \right) \beta \left(\lambda_{t+1} \right)^{-\gamma} \right] = 0$$

This result follows directly from the Euler equation in (17). It has important implications for empirical work in asset pricing. First, in spite of the market incompleteness and binding solvency constraints, an econometrician can estimate the coefficient of risk aversion directly from aggregate consumption data and the excess return on stocks, as in Hansen and Singleton (1982).[22] Second, it provides a strong justification for explaining the cross-section of excess returns using the CCAPM, without trying to match the risk-free rate.

6.2 Risk Premia

Not surprisingly, the equilibrium risk premium is identical to the one in the representative agent economy.[23] Since the risk-free rate is higher than in the Arrow and *HL* model, the price of the stock is correspondingly lower. However, the multiplicative risk premium is the same in all three models and it is constant.

The stochastic discount factors that prices stochastic payoffs in the representative agent economy and the Arrow economy only differ by a non-random multiplicative term, equal to the ratio of (growth-deflated) risk-free interest rates in the two models. We use the superscript *RE* to denote the Representative Agent economy.

Proposition 6.1. *In the Arrow economy, there is a unique SDF given by*

$$m^A_{t+1} = m^{RE}_{t+1} \kappa_t$$

[22] In the bond economy, we can only use return data for a claim to aggregate consumption.

[23] This does not immediately follow from Lemma 6.1.

with a non-random multiplicative term given by:

$$\kappa_t = \frac{\hat{R}_t^{RE}}{\hat{R}_t^A} \geq 1$$

Note that the term κ_t is straightforward to compute since $\hat{R}_t^A$ equals the interest rate in the Bewley model discussed in section (3). What about the *HL* economy? We have shown, using the equivalence result between the Arrow and the *HL* model, that the Arrow economy's stochastic discount factor m_{t+1}^A is also a valid stochastic discount factor in the *HL* economy. So, our results below carry over to the *HL* economy as well.

The proof that risk premia are unchanged between the representative agent model and the Arrow model now follows directly from the previous decomposition of the SDF.[24] Let $R_{t,j}\left[\{d_{t+k}\}\right]$ denote the j-period holding return on claim to $\{d_{t+k}\}$. Consequently $R_{t,1}[1]$ is the gross risk-free rate and $R_{t,1}[\alpha e_{t+k}]$ is the one-period holding return on a k-period strip of the aggregate endowment (a claim to α times the aggregate endowment k periods from now). With this notation in place, we can state our main result.

Theorem 6.1. *The multiplicative risk premium in the Arrow economy equals that in the representative agent economy*

$$1 + \nu_t^A = 1 + \nu_t^{RE} = \frac{E_t R_{t,1}\left[\{e_{t+k}\}\right]}{R_{t,1}[1]}$$

Thus, the extent to which households smooth idiosyncratic income shocks (in the Arrow model or in the *HL* model) amount has absolutely no effect on the size of the risk premia; it merely lowers the risk-free rate. Exactly the same result applies to the *HL* model as well[25] The market incompleteness does not generate any dynamics in the conditional risk premia either: the conditional risk premium is constant.

[24]The proof strategy follows Alvarez and Jermann (2001) who derive a similar result in the context of a *complete markets* model populated by two agents that face endogenous solvency constraints.

[25] Cochrane and Hansen (1992) had already established a similar aggregation result for the case in which households face market wealth constraints, but in a complete markets environment. We show this result survives even if households trade only a stock and a bond.

7 Robustness and Extensions of the Main Results

In this section, we investigate how robust our results are, and we demonstrate that our assumption that the aggregate shocks are *i.i.d* over time, which means the growth rates of the aggregate endowment are *i.i.d* over time, is not crucial for our results. We conclude by discussing the effects of non-time-separable utility and preference heterogeneity.

7.1 Non-iid Aggregate Shocks

When the aggregate shocks z follow an arbitrary, finite state, Markov chain, the growth-adjusted time discount factor $\hat{\beta}(z)$ depends on the current aggregate state, and, as a result, the aggregate endowment shock acts as an aggregate taste shock in the deflated economy. This shock renders all households more or less impatient. Of course, households are not able to insure against this shock at all, since it affects all households in the same way. As a consequence, non-iid aggregate endowment shocks, acting as taste shocks in the deflated economy, only affect the price/dividend ratio and the interest rate, but not the risk premium. However, in this case, there is trade in the Arrow securities market.

Stationary Bewley economy In order to establish these results, as before we use a stationary Bewley economy as a computational device. Denote by $\hat{\Pi}(z')$ the invariant distribution over the aggregate shock under the hatted measure. Agents in the stationary economy discount future utility flows using the deterministic time discount factor process $\left\{\widetilde{\beta}_t\right\}$, where $\widetilde{\beta}_t$ is given by the average subjective discount factor:

$$\widetilde{\beta}_t = \sum_{z_0} \hat{\Pi}(z_0) \sum_{z^{t-1}} \widehat{\phi}(z^{t-1}|z_0)\hat{\beta}_{0,t-1}(z^{t-1}|z_0), t \geq 1, \tag{24}$$

and where we have used $\hat{\beta}_{0,\tau}(z^\tau|z_0)$ to denote the product of time discount factors:

$$\hat{\beta}_{0,\tau}(z^\tau|z_0) = \hat{\beta}(z_0)\hat{\beta}(z_1)\ldots\hat{\beta}(z_\tau).$$

Given this non-random sequence of subjective time discount factors, we can define an equilibrium for the Bewley economy in the standard way. In the simplest case of i.i.d shocks we obtain $\widetilde{\beta}_t = \hat{\beta}^t$, as in the case we discussed in the previous section.

We choose this particular sequence of subjective time discount factors because it ensures that the time zero budget constraint in the deflated Arrow economy is satisfied given the same initial wealth distribution Θ_0 as in the stationary Bewley economy. In order to construct equilibrium allocations, we first determine equilibrium allocations and interest rates in the Bewley economy with time discount factors $\left\{\widetilde{\beta}_t\right\}$, analogous to the analysis in section (3). In a second step, we make these allocations and interest rates into an equilibrium of the *actual* Arrow economy with time-varying discount factors by adjusting the risk-free interest rate in proportion to the taste shock $\hat{\beta}(z)$. We do not change the allocations, and we back out the implied Arrow securities positions. To understand the effect of these aggregate taste shocks on the time discount rate in the Bewley economy, we consider a simple example.

Example 1. *If the $\hat{\beta}(z)$ are lognormal and i.i.d with variance σ^2 , then the effective average time discount rate between time 0 and time t is given by:*

$$\frac{\widetilde{\rho}_t}{t} = \widehat{\rho} - \frac{1}{2}\sigma^2 \text{ for any } t \geq 1$$

where $\widetilde{\beta}_t = e^{-\widetilde{\rho}_t}$ and $E\hat{\beta}(z) = e^{-\widehat{\rho}}$. As a result, the time discount rate in the Bewley economy is lower than the actual discount rate, because of the risk associated with the taste shocks.

We use $\{\hat{c}_t(\theta_0, y^t), \hat{a}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$ and prices $\left\{\hat{R}_t, \hat{v}_t\right\}$ to denote the Bewley equilibrium for given $\left\{\widetilde{\beta}_t\right\}$. Only the total net wealth positions in the Bewley economy are pinned down (since the bond and the stock, by a no-arbitrage condition, have exactly the same return, and are both risk-free assets):

$$\hat{b}_t(\theta_0, y^t) = \hat{a}_t(\theta_0, y^t) + \hat{\sigma}_t(\theta_0, y^t)(\hat{v}_t + \alpha).$$

Without loss of generality, we focus on the case where $\hat{a}_t(\theta_0, y^t) = 0$ for all y^t. Our goal is to show that the consumption and share allocation $\{\hat{c}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$ can be made into an Arrow equilibrium, and then to argue why we need to choose the specific discount factor sequence in (24) for the Bewley model. To fix notation, let

$$\tilde{Q}_{t,\tau} =_{j=0}^{\tau-t-1} \hat{R}_{t+j}^{-1} = \frac{1}{\widehat{R}_{\tau,t}}$$

denote the price (in the Bewley equilibrium) for one unit of consumption to be delivered at time τ, in terms of consumption at time t. Also, let us use the convention $\tilde{Q}_\tau = \tilde{Q}_{t=0,\tau}$ and $\tilde{Q}_{t=\tau} = 1$.

Arrow economy Next, we follow the same recipe as before, by implementing the Bewley allocations as part of an equilibrium for the deflated Arrow economy, but in this case, this will inolve trade in the contingent claims market. To start off, we conjecture that the Arrow-Debreu prices in the deflated Arrow economy are given by:

$$\hat{Q}(z^t|z_0) = \frac{\hat{\beta}_{0,t-1}(z^{t-1})\hat{\phi}(z^t|z_0)}{\widetilde{\beta}_t \hat{R}_{0,t}},$$

from which we can easily recover the Arrow prices from the usual relationship:

$$\frac{\hat{Q}(z^{t+1}|z_0)}{\hat{Q}(z^t|z_0)} = q(z_{t+1}, z_t) = \hat{\beta}(z_t)\hat{\phi}(z_{t+1}|z_t)\frac{1}{\hat{R}_t}\frac{\widetilde{\beta}_t}{\widetilde{\beta}_{t+1}} \tag{25}$$

where $\frac{1}{\hat{R}_t} = \frac{\hat{R}_{0,t}}{\widehat{R}_{0,t+1}}$. These Arrow prices are Markovian, since $\hat{R}_t$ and $\widetilde{\beta}_t$ are deterministic.

Lemma 7.1. *The household Euler equations are satisfied in the Arrow model at the Bewley allocations $\{\hat{c}_{t+1}(y^t, y_{t+1})\}$ and Arrow prices $\{q(z_{t+1}, z_t)\}$ given by (25).*

Naturally, this implies that the state-contingent interest rates in the Arrow model are given by

$$\frac{1}{\hat{R}_t^A(z_t)} = \hat{\beta}(z_t)\frac{\widetilde{\beta}_t}{\hat{R}_t \widetilde{\beta}_{t+1}},$$

which can easily be verified from equation (25).

Trading Trading starts before the initial aggregate shock z_0 is observed. In the Arrow economy, there is trade in the contingent claims markets. First, we need to check that these contingent claims positions implied by the Bewley allocations clear the securities markets.

Proposition 7.1. *The contingent claims positions implied by the Bewley allocations:*

$$\begin{aligned}\hat{a}_{t-1}(y^{t-1}, z^t, \theta_0) &= \hat{c}_t(y^t, \theta_0) - \eta(y_t) + \sum_{\tau=t+1}^{\infty} \sum_{z^\tau, y^\tau} \hat{Q}_t(z^\tau | z_t) \left(\hat{c}_\tau(y^\tau, \theta_0) - \eta(y_\tau)\right) \\ &\quad -\sigma_{t-1}(y^{t-1}) \left[\hat{v}_t(z_t) + \alpha\right] = \hat{a}_{t-1}(y^{t-1}, z_t, \theta_0) \end{aligned} \tag{26}$$

clear the bond market

The contingent claims positions are used to hedge against interest rate shocks, as is clear from this restatement of the bond position:

$$\begin{aligned}\hat{a}_{t-1}(y^{t-1}, z_t, \theta_0) &= \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \left(\hat{Q}_\tau(z^\tau | z_t) - \widetilde{Q}_\tau\right) \sum_{\eta^\tau} \left(\hat{c}_\tau(y^\tau, \theta_0) - \eta(y_\tau)\right) \\ &\quad -\sigma_{t-1}(y^{t-1})\alpha \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \left(\hat{Q}_\tau(z^\tau | z_t) - \widetilde{Q}_\tau\right)\end{aligned}$$

The difference between $\hat{Q}_\tau(z^\tau | z_t)$ and $\widetilde{Q}$ is governed by the interest rate shocks. If the aggregate consumption growth shocks are i.i.d, this gap is zero. Second, to close our argument, we need to make sure that no wealth transfers are required to implement the Bewley equilibrium allocations in the deflated Arrow economy. In other words, we need to make sure that the state contingent bond portfolio at time zero, before the realization of z_0, is worth exactly zero. For this we proceed in two steps. First, we show that at time zero the state prices in the Arrow and Bewley economy coincide.

Lemma 7.2. *As a result of our choice of $\left\{\widetilde{\beta}_t\right\}$ in the Bewley economy, on average these AD prices coincide with the Bewley prices at time 0, before the realization of z_0:*

$$\sum_{z_0} \hat{\Pi}(z_0) \sum_{z^\tau} \hat{Q}_\tau(z^\tau | z_0) = \tilde{Q}_{0,\tau},$$

Second, using the result about prices, we show that the cost of the bond portfolio is zero.

Lemma 7.3. *The cost at time 0 of the time 0 state-contingent claims portfolio is zero:*

$$\sum_{z^t} \hat{\Pi}(z_0)\left[\hat{a}_{-1}(y^{-1}, z_0, \theta_0)\right] = 0$$

Since we know the contingent bond positions are zero cost, the time 0 budget constraint is satisfied in the deflated Arrow economy for the exact same initial wealth distribution Θ_0 as in the Bewley economy.

Theorem 7.1. *An equilibrium of a stationary Bewley economy, populated by households with constant discount factor* $\widetilde{\beta}$, $\{\hat{c}_t(\theta_0, y^t), \hat{a}_t(\theta_0, y^t) = 0,\ \hat{\sigma}_t(\theta_0, y^t)\}$ *and* $\{\hat{R}_t, \hat{v}_t\}$ *can be made into an equilibrium for the Arrow economy with growth,* $\{a_t(\theta_0, s^t, z_{t+1})\}$, $\{\sigma_t(\theta_0, s^t)\}$, $\{c_t(\theta_0, s^t)\}$ *and* $\{q_t(z^t, z_{t+1})\}$, $\{v_t(z^t)\}$, *with*

$$\begin{aligned}
c_t(\theta_0, s^t) &= \hat{c}_t(\theta_0, y^t) \\
\sigma_t(\theta_0, s^t) &= \hat{\sigma}_t(\theta_0, y^t) \\
a_t(\theta_0, s^t, z_{t+1}) &= \hat{a}_{t-1}(y^{t-1}, z_t, \theta_0)e_t(z^t) \text{ defined above} \\
v_t(z_t) &= \sum_{z_{t+1}} \frac{\hat{\phi}(z_{t+1}|z_t)}{\lambda(z_{t+1})}\left[\frac{v_{t+1}(z_{t+1}) + \alpha e_{t+1}(z_{t+1})}{\hat{R}^A_t(z_t)}\right] \\
\hat{R}^A_t(z_t) &= \frac{\hat{R}_t\widetilde{\beta}_{t+1}}{\hat{\beta}(z_t)\widetilde{\beta}_t} \\
q_t(z^t, z_{t+1}) &= \frac{1}{\hat{R}^A_t(z_t)} * \frac{\phi(z_{t+1}|z_t)\lambda(z_{t+1})^{-\gamma}}{\sum_{z_{t+1}}\phi(z_{t+1}|z_t)\lambda(z_{t+1})^{1-\gamma}}
\end{aligned}$$

Evidently, as before one can back out the Arrow prices for the deflated economy from the prices in the deflated Bewley economy. The contingent claims positions required to implement the Bewley allocations where determined above.

Risk Premia Of course, this implies that our baseline irrelevance result for risk premia survives the introduction of non-i.i.d. aggregate shocks, provided that a complete menu of aggregate-state-contingent securities is traded. These aggregate taste shocks only affect interest rates and price/dividend ratios, not risk premia. When agents in the transformed economy become more impatient, the interest rises and the price/dividend ratio decreases, but the conditional expected excess return is unchanged.

Solvency Constraints So far we have abstracted from solvency constraints. Remember that we originally assumed that the solvency constraints satisfy $K_t(s^t) = \hat{K}_t(y^t)e_t(z^t)$ and $M_{t+1}(s^{t+1}) = \hat{M}_t(y^t)e_t(z^{t+1})$. The allocations computed in the stationary economy using $\hat{K}_t(y^t)$ and $\hat{M}_t(y^t)$ as solvency constraints, satisfy a modified version of the solvency constraints $K_t(s^t)$ and $M_{t+1}(s^{t+1})$.

Lemma 7.4. *The allocations from the ergodic economy satisfy the modified solvency constraints:*

$$\begin{aligned} K_t^*(s^t) &= K_t(s^t) - \sum_{z_{t+1}} q_t(z^t, z_{t+1}) a_t(s^t, z_{t+1}) \\ M_{t+1}^*(s^{t+1}) &= M_{t+1}(s^{t+1}) - a_t(s^t, z_{t+1}) \end{aligned}$$

This means that in the case of non-i.i.d. aggregate shocks our implementation and asset pricing results is not quite robust to the introduction of binding solvency constraints: if the allocations satisfy the constraints in the stationary Bewley economy, they satisfy the modified solvency constraints in the actual Arrow economy, but not the ones we originally specified, because of the nonzero state-contingent claims positions. However, it is easy to verify using the result in Corollary (7.3) in the appendix that these modified solvency constraints coincide with the actual ones on average (averaged across z shocks). In addition, the violations are likely to be small for plausible calibrations of the aggregate endowment growth process, because the interest rate would not vary too much over the business cycle. Furthermore, these deviations are completely due to the impact of interest rate changes on the value of the portfolio. The risk premia are still constant over the business cycle.

***HL* Economy** For the *HL* model, the same equivalence result obviously no longer holds, because the market for state contingent Arrow securities is now operative in the Arrow economy. When there is predictability in aggregate consumption growth, households actually find it optimal to trade the state-contingent Arrow securities, but the market structure in the *HL* economy prevents them from doing so.

7.2 Preferences

What role do preferences play in our results? Well, it is key to have homogeneous preferences. Time separability is not critical. In section (A.2) of the appendix, we study the case of Epstein-Zin preferences. In the case of i.i.d. aggregate consumption growth, the irrelevance result survives.

When different groups of agents have different risk aversion coefficients γ, the Arrow economy behaves like a complete markets economy with N different agents[26], exactly as analyzed by Dumas (1989). Following Constantinides (1982), we can construct a mongrel utility function for pricing assets directly off aggregate consumption growth. Preference heterogeneity does introduce time-variation into risk premia. But, again, the market incompleteness itself does not contribute anything to the risk premia beyond what one obtains in the complete markets case. However, there is trade in the contingent claims markets in equilibrium and the equivalence with the *HL* economy breaks down.

8 Related Literature and Conclusion

The results of our paper make contact with the literature on aggregation. Constantinides (1982), building on work by Negishi (1960) and Wilson (1968), derives an aggregation result for heterogenous agents in complete market models, implying that assets can be priced off the intertemporal marginal rate of substitution of an agent who consumes the aggregate endowment. We extend his result to a large class of incomplete market models with idiosyncratic income shocks.

Most of the work on incomplete markets and risk premia documents the moments of model-generated data for particular calibrations, but there are few analytical results. Levine and Zame (2002) show that in economies populated by agents with infinite hori-

[26]The proof is available upon request.

zons, the equilibrium allocations in the limit, as their discount factors go to one, converge to the complete markets allocations. Consequently the pricing implications of the incomplete markets model converge to that of the representative agent model as households become perfectly patient. We provide a qualitatively similar equivalence result that applies only to the risk premium. The result, however, does not depend on the time discount factor of households. For households with CARA utility, closed form solutions of the individual decision problem in incomplete markets models with idiosyncratic risk are sometimes available, as Willen (1999) shows.[27] We use CRRA preferences for our results, and we obtain an unambiguous (and negative) result for the impact of uninsurable income risk for the equity premium in the case that the distribution of individual income shocks is independent of aggregate shocks. Finally, Lettau (2006) shows that if household consumption (in logs) consists of an aggregate and an idiosyncratic part, the latter does not affect risk premia. We show that this characterization of the household consumption process is indeed the correct one in equilibrium in a large class of incomplete market models with potentially binding borrowing constraints.

Most related to our study is the work by Constantinides and Duffie (1996), who consider an environment in which agents face permanent, idiosyncratic income shocks and can trade stocks and bonds. Their equilibrium is characterized by no trade in financial markets. By choosing the right stochastic income process, CD deliver equilibrium asset prices with all desired properties. Krebs (2005) extends this result to a production economy. In his world, as in ours, the wealth distribution is not required as a state variable to characterize equilibria, in spite of the presence of aggregate shocks, but, as in CD (1996), the equilibrium is autarkic, so that households cannot diversify any of their risk. In our models households will be able to smooth some of their income shocks, and the equilibrium features trade in assets, but we can still fully characterize equilibrium asset prices.

Finally, our paper establishes the existence of a recursive competitive equilibrium with only asset holdings in the state space, albeit under a transformed probability measure. Kubler and Schmedders (2002) establish the existence of such an equilibrium, but only under very strong conditions. Miao (2004) relaxes these conditions, but he includes con-

[27]Eliminating wealth effects simplifies the analysis.

tinuation utilities in the state space.

In ongoing work we investigate whether our analytical results can be used to derive explicit results or bounds for the effect of uninsurable idiosyncratic income risk on the equity premium if the variance of this risk is allowed to vary over the cycle, as suggested by Storesletten, Telmer and Yaron (2004). But since an analytical implementation result that allowed us to construct equilibrium allocations in the Arrow and HL model from allocations in the Bewley model is not available, the results of this analysis are likely going to be computational in nature.

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A Additional Definitions

A.1 Recursive Competitive Equilibrium

As stated above, without loss of generality we can abstract from trade in bonds. Let $A \subset \mathbf{R}$ denote the set of possible share holdings $\mathcal{B}(A)$ the Borel $\sigma-$algebra of A. Y is the set of possible labor income draws and $\mathcal{P}(Y)$ is the power-set of Y. Define the state space as $Z = A \times Y$ and $\mathcal{B}(Z) = \mathcal{P}(Y) \times \mathcal{B}(A)$. We define the households' recursive problem for the case of a stationary recursive equilibrium in which the cross-sectional distribution Φ and prices are constant:

$$\begin{aligned}
v(\sigma, y) &= \max_{c \geq 0, \sigma'} \left\{ u(c) + \hat{\beta} \sum_{y'} \varphi(y'|y) v(\sigma', y') \right\} \\
&\quad \text{s.t.} \\
c + \sigma' \hat{v} &= y + \sigma \left[\hat{v} + \alpha \right] \\
\sigma' \hat{v} &\geq \hat{K} \text{ or } \sigma' \left[\hat{v} + \alpha \right] \geq \hat{M}
\end{aligned}$$

Definition A.1. *A stationary recursive competitive equilibrium (RCE) for the Bewley economy is a value function $v : Z \to R$, policy functions for the household $\sigma' : Z \to R$ and $c : Z \to R$, a stock price $\hat{v}$ and a measure Φ such that*

1. *v, σ, c' are measurable with respect to $\mathcal{B}(Z)$, v satisfies the household's Bellman equation and σ', c are the associated policy functions, given $\hat{v}$*

2. *Market clearing*

$$\begin{aligned}
\int c d\Phi &= 1 \\
\int \sigma' d\Phi &= 1
\end{aligned}$$

3. *The distribution is stationary*

$$\Phi(\mathcal{A}, \mathcal{Y}) = \int Q((a, y), (\mathcal{A}, \mathcal{Y})) \Phi(da \times dy)$$

where Q is the Markov transition function induced by the exogenous Markov chain $\hat{\phi}$ and the decision rule σ'.

Existence of a stationary RCE in the Bewley economy can now be guaranteed under the standard conditions (see Huggett (1993) and Aiyagari (1994) (Aiyagari 1994)). More importantly, the standard algorithm to compute stationary equilibria in Bewley models can be applied.

Algorithm A.1. *1. Guess a price $\hat{v}$.*

2. Solve the recursive household problem.

3. Find the stationary measure Φ associated with the household decision rule σ and the exogenous Markov chain φ.

4. Compute the excess demand in the stock market,

$$d(\hat{v}) = \int \sigma' d\Phi - 1$$

5. If $d(\hat{v}) = 0$, we are done, if not, adjust the guess for $\hat{v}$ and repeat 2.-5.

Remark 1. *Once the equilibrium stock price $\hat{v}$ is determined, we can easily derive the equilibrium interest rate as*

$$\hat{R} = \frac{\hat{v} + \alpha}{\hat{v}}$$

A.2 Arrow Economy

A.2.1 Market Structure and Equilibrium

Let $a_t(s^t, z_{t+1})$ denote the quantity purchased of a security that pays off one unit of the consumption good if aggregate shock in the next period is z_{t+1}, irrespective of the idiosyncratic shock y_{t+1}. Its price today is given by $q_t(z^t, z_{t+1})$. In addition, households trade shares in the Lucas tree. We use $\sigma_t(s^t)$ to denote the number of shares a household with history $s^t = (y^t, z^t)$ purchases today and we let $v_t(z^t)$ denote the price of one share.

An agent starting period t with initial wealth $\theta_t(s^t)$ buys consumption commodities in the spot market and trades securities subject to the usual budget constraint:

$$c_t(s^t) + \sum_{z_{t+1}} a_t(s^t, z_{t+1}) q_t(z^t, z_{t+1}) + \sigma_t(s^t) v_t(z^t) \leq \theta_t(s^t). \tag{27}$$

If next period's state is $s^{t+1} = (s^t, y_{t+1}, z_{t+1})$, her wealth is given by her labor income, the payoff from the contingent claim purchased in the previous period as well as the value of her position on the stock, including dividends:

$$\begin{aligned} \theta_{t+1}(s^{t+1}) &= \underbrace{\eta(y_{t+1}, z_{t+1}) e_{t+1}(z_{t+1})}_{\textit{labor income}} + \underbrace{a_t(s^t, z_{t+1})}_{\textit{contingent payoff}} \\ &\quad + \underbrace{\sigma_t(s^t) \left[v_{t+1}(z^{t+1}) + \alpha e_{t+1}(z_{t+1}) \right]}_{\textit{value of shares in Lucas tree}} \end{aligned} \tag{28}$$

The definition of an equilibrium in this economy is standard. Each household is assigned a label that consists of its initial financial wealth θ_0 and its initial state $s_0 = (y_0, z_0)$. A household of type (θ_0, s_0) chooses consumption allocations $\{c_t(\theta_0, s^t)\}$, trading strategies for Arrow securities $\{a_t(\theta_0, s^t, z_{t+1})\}$ and shares $\{\sigma_t(\theta_0, s^t)\}$ to maximize her expected utility (1), subject to the budget constraints (27) and subject to solvency constraints (10) or (11). We are ready to define an Arrow equilibrium.[28]

Definition A.2. *For initial aggregate state z_0 and distribution Θ_0 over (θ_0, y_0), a competitive equilibrium for the Arrow economy consists of household allocations $\{a_t(\theta_0, s^t, z_{t+1})\}$, $\{\sigma_t(\theta_0, s^t)\}$, $\{c_t(\theta_0, s^t)\}$ and prices $\{q_t(z^t, z_{t+1})\}$, $\{v_t(z^t)\}$ such that*

1. *Given prices, household allocations solve the household maximization problem*

[28] Several elements are worth commenting on. First, since there exists a full set of Arrow securities spanning the aggregate uncertainty, the stock is a redundant asset and we could have formulated the household problem and thus the equilibrium definition without explicitly mentioning the stock. Incorporating the stock explicitly, though, will make our asset pricing and equivalence results to follow clearer. Second, by Walras' law the goods market clearing condition (or one of the asset market clearing conditions) is redundant.

2. The goods market clears for all z^t,

$$\int \sum_{y^t} \frac{\pi(y^t, z^t|y_0, z_0)}{\pi(z^t|z_0)} c_t(\theta_0, s^t) d\Theta_0 = e_t(z^t)$$

3. The asset markets clear for all z^t

$$\begin{aligned}
\int \sum_{y^t} \frac{\pi(y^t, z^t|y_0, z_0)}{\pi(z^t|z_0)} \sigma_t(\theta_0, s^t) d\Theta_0 &= 1 \\
\int \sum_{y^t} \frac{\pi(y^t, z^t|y_0, z_0)}{\pi(z^t|z_0)} a_t(\theta_0, s^t, z_{t+1}) d\Theta_0 &= 0 \text{ for all } z_{t+1} \in Z
\end{aligned}$$

A.3 Optimality Conditions for Arrow Economy

Finally we list the optimality conditions. Attaching Lagrange multiplier $\mu_t(s^t) \geq .0$ to the constraint (13) and $\kappa_t(s^t, z_{t+1}) \geq 0$ to the constraint (14), the Euler equations of the detrended Arrow economy are given by:

$$\begin{aligned}
1 &= \frac{\hat{\beta}(s_t)}{\hat{q}_t(z^t, z_{t+1})} \sum_{s_{t+1}} \hat{\pi}(s_{t+1}|s_t) \frac{u'(\hat{c}_{t+1}(s^t, s_{t+1}))}{u'(\hat{c}_t(s^t))} && (29) \\
&+ \mu_t(s^t) + \frac{\kappa_t(s^t, z_{t+1})}{\hat{q}_t(z^t, z_{t+1})} \ \forall z_{t+1}. && (30) \\
1 &= \hat{\beta}(s_t) \sum_{s_{t+1}} \hat{\pi}(s_{t+1}|s_t) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)} \right] \frac{u'(\hat{c}_{t+1}(s^t, s_{t+1}))}{u'(\hat{c}_t(s^t))} \\
&+ \mu_t(s^t) + \sum_{z_{t+1}} \kappa_t(s^t, z_{t+1}) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)} \right]. && (31)
\end{aligned}$$

Only one of the two Lagrange multiplier is relevant, depending on which version of the shortsale constraint we are considering. Finally, the complementary slackness conditions for the Lagrange multipliers are given by

$$\begin{aligned}
\mu_t(s^t) \left[\sum_{z_{t+1}} \hat{a}_t(s^t, z_{t+1}) \hat{q}_t(z^t, z_{t+1}) + \sigma_t(s^t) \hat{v}_t(z^t) - \hat{K}_t(y^t) \right] &= 0 \\
\kappa_t(s^t, z_{t+1}) \left[\hat{a}_t(s^t, z_{t+1}) + \sigma_t(s^t) \left[\hat{v}_{t+1}(z^{t+1}) + \alpha \right] - \hat{M}_t(y^t) \right] &= 0
\end{aligned}$$

The first order conditions and complementary slackness conditions, together with the appropriate transversality condition, are necessary and sufficient conditions for optimality on the household side.

In addition, we list the appropriate transversality conditions:

$$\lim_{t\to\infty}\sum_{s^t}\hat{\beta}^t\hat{\pi}(s^t|s_0)u'(c_{t+1}(y^{t+1},z^{t+1}))[\hat{a}_t(s^t,z_{t+1})-\hat{M}_t(y^t)]=0,$$

and

$$\lim_{t\to\infty}\sum_{s^t}\hat{\beta}^t\hat{\pi}(s^t|s_0)u'(c_t(y^t,z^t))[\hat{a}_t(s^t)-\hat{K}_t(y^t)]=0.$$

A.4 *HL* Economy

Wealth tomorrow in state $s^{t+1}=(s^t,y_{t+1},z_{t+1})$ is given by

$$\begin{aligned}\theta_{t+1}(s^{t+1}) &= \underbrace{\eta(y_{t+1})e_{t+1}(z_{t+1})}_{\textit{labor income}}+\underbrace{b_t(s^t)}_{\textit{bond payoff}} \\ &\quad +\underbrace{\sigma_t(s^t)\left[v_{t+1}(z^{t+1})+\alpha e_{t+1}(z_{t+1})\right]}_{\textit{value of shares in Lucas tree}}.\end{aligned} \tag{32}$$

Definition A.3. *For initial aggregate state z_0 and distribution Θ_0 over (θ_0,y_0), a competitive equilibrium for the* HL *economy consists of trading strategies $\left\{b_t(\theta_0,s^t)\right\}$, $\left\{c_t(\theta_0,s^t)\right\}$, $\left\{\sigma_t(\theta_0,s^t)\right\}$ and interest rates $\left\{R_t(z^t)\right\}$ and share prices $\left\{v_t(z^t)\right\}$ such that*

1. *Given prices, allocations solve the household maximization problem*

2. *The goods market clears*

$$\int\sum_{y^t}\frac{\pi(y^t,z^t|y_0,z_0)}{\pi(z^t|z_0)}c_t(\theta_0,s^t)d\Theta_0=e_t(z^t)$$

3. The asset markets clear for all z^t

$$\begin{aligned}\int \sum_{y^t} \frac{\pi(y^t, z^t|y_0, z_0)}{\pi(z^t|z_0)} \sigma_t(\theta_0, s^t) d\Theta_0 &= 1 \\ \int \sum_{y^t} \frac{\pi(y^t, z^t|y_0, z_0)}{\pi(z^t|z_0)} b_t(\theta_0, s^t) d\Theta_0 &= 0.\end{aligned}$$

In the detrended *HL* economy the Euler equations read as

$$\begin{aligned}1 &= \hat{\beta}(s_t) \sum_{s_{t+1}} \hat{\pi}(s_{t+1}|s_t) \left[\frac{R_t(z^t)}{\lambda(z_{t+1})}\right] \frac{u'(\hat{c}_{t+1}(s^t, s_{t+1}))}{u'(\hat{c}_t(s^t))} \\ &\quad + \mu_t(s^t) + \sum_{z_{t+1}} \kappa_t(s^t, z_{t+1}) \left[\frac{R_t(z^t)}{\lambda(z_{t+1})}\right] \qquad (33) \\ &= \hat{\beta}(s_t) \sum_{s_{t+1}} \hat{\pi}(s_{t+1}|s_t) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)}\right] \frac{u'(\hat{c}_{t+1}(s^t, s_{t+1}))}{u'(\hat{c}_t(s^t))} \\ &\quad + \mu_t(s^t) + \sum_{z_{t+1}} \kappa_t(s^t, z_{t+1}) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)}\right], \qquad (34)\end{aligned}$$

with complementary slackness conditions given by:

$$\begin{aligned}\mu_t(s^t) \left[\frac{\hat{b}_t(s^t)}{R_t(z^t)} + \sigma_t(s^t)\hat{v}_t(z^t) - \hat{K}_t(s^t)\right] &= 0 \\ \kappa_t(s^t, z_{t+1}) \left[\frac{\hat{b}_t(s^t)}{\lambda(z_{t+1})} + \sigma_t(s^t)\left[\hat{v}_{t+1}(z^{t+1}) + \alpha\right] - \hat{M}_t(s^t, z_{t+1})\right] &= 0\end{aligned}$$

In addition, we list the appropriate transversality conditions:

$$\lim_{t\to\infty} \sum_{s^t} \hat{\beta}^t \hat{\pi}(s^t|s_0) u'(\hat{c}_t(y^t, z^t)) \left[\frac{\hat{b}_t(s^t)}{R_t(z^t)}\right] = 0.$$

$$\lim_{t\to\infty} \sum_{s^t} \hat{\beta}^t \hat{\pi}(s^t|s_0) u'(\hat{c}_t(y^t, z^t)) \sigma_{t-1}(s^{t-1}) \left[\hat{v}_t(z^t) + \alpha\right] = 0.$$

A.5 Representative Agent Model

The budget constraint reads as

$$\begin{aligned} & c_t(z^t) + \sum_{z_{t+1}} a_t(z^t, z_{t+1}) q_t(z^t, z_{t+1}) + \sigma_t(z^t) v_t(z^t) \\ \leq \; & e_t(z^t) + a_{t-1}(z^{t-1}, z_t) + \sigma_{t-1}(z^{t-1}) \left[v_t(z^t) + \alpha e_t(z_t) \right] \end{aligned}$$

After deflating by the aggregate endowment $e_t(z^t)$, the budget constraint reads as

$$\begin{aligned} & \hat{c}_t(z^t) + \sum_{z_{t+1}} \hat{a}_t(z^t, z_{t+1}) \hat{q}_t(z^t, z_{t+1}) + \sigma_t(z^t) \hat{v}_t(z^t) \\ \leq \; & 1 + \hat{a}_{t-1}(z^{t-1}, z_t) + \sigma_{t-1}(z^{t-1}) \left[\hat{v}_t(z^t) + \alpha \right], \end{aligned}$$

where $\hat{a}_t(z^t, z_{t+1}) = \frac{a_t(z^t, z_{t+1})}{e_{t+1}(z^{t+1})}$ and $\hat{q}_t(z^t, z_{t+1}) = q_t(z^t, z_{t+1}) \lambda(z_{t+1})$ as well as $\hat{v}_t(z^t) = \frac{v_t(z^t)}{e_t(z^t)}$, as before in the Arrow model. Obviously, in an equilibrium of this model the representative agent consumes the aggregate endowment. Asset pricing in this economy is very simple.

Lemma A.1. *Equilibrium asset prices are given by*

$$\begin{aligned} \hat{q}_t(z^t, z_{t+1}) &= \hat{\beta} \hat{\phi}(z_{t+1}) \text{ for all } z_{t+1} \\ \hat{v}_t(z^t) &= \hat{\beta} \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) \left[\hat{v}_{t+1}(z^{t+1}) + \alpha \right] \end{aligned} \tag{35}$$

A.6 Recursive Utility

We consider the class of preferences due to Epstein and Zin (1989) . Let $V(c^i)$ denote the utility derived from consuming c^i :

$$V(c^i) = \left[(1-\beta) c_t^{1-\rho} + \beta (\mathcal{R}_t V_1)^{1-\rho} \right]^{\frac{1}{1-\rho}}, \tag{36}$$

where the risk-adjusted expectation operator is defined as:

$$\mathcal{R}_t V_{t+1} = \left(E_t V_{t+1}^{1-\alpha} \right)^{1/1-\alpha}.$$

α governs risk aversion and ρ governs the willingness to substitute consumption intertemporally. These preferences impute a concern for the timing of the resolution of uncertainty to agents. In the special case where $\rho = \frac{1}{\alpha}$, these preferences collapse to standard power utility preferences with CRRA coefficient α. As before, we can define *growth-adjusted* probabilities and the growth-adjusted discount factor as:

$$\begin{aligned}
\hat{\pi}(s_{t+1}|s_t) &= \frac{\pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}}{\sum_{s_{t+1}} \pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}} \\
\text{and } \hat{\beta}(s_t) &= \beta \left(\sum_{s_{t+1}} \pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha} \right)^{\frac{1-\rho}{1-\alpha}}
\end{aligned}$$

As before, $\hat{\beta}(s_t)$ is stochastic as long as the original Markov process is not *iid* over time. Note that the adjustment of the discount rate is affected by both ρ and α. If $\rho = \frac{1}{\alpha}$, this transformation reduces to the case we discussed in section (2).

Finally, let $\hat{V}_t(\hat{c})(s^t)$ denote the lifetime expected continuation utility in node s^t, under the new transition probabilities and discount factor, defined over consumption shares $\left\{\hat{c}_t(s^t)\right\}$:

$$\hat{V}_t(\hat{c})(s^t) = \left[(1-\beta)\hat{c}_t^{1-\rho} + \hat{\beta}(s_t)(\hat{\mathcal{R}}_t \hat{V}_{t+1}(s^{t+1}))^{1-\rho}\right]^{\frac{1}{1-\rho}},$$

where $\mathcal{R}$ denotes the following operator:

$$\hat{\mathcal{R}}_t V_{t+1} = \left(\hat{E}_t \hat{V}_{t+1}^{1-\alpha}\right)^{1/1-\alpha}.$$

and $\hat{E}$ denotes the expectation operator under the hatted measure $\hat{\pi}$.

Proposition A.1. *Households rank consumption share allocations in the de-trended economy in exactly the same way as they rank the corresponding consumption allocations in the original growing economy: for any s^t and any two consumption allocations c, c'*

$$V(c)(s^t) \geq V(c')(s^t) \Longleftrightarrow \hat{V}(\hat{c})(s^t) \geq \hat{V}(\hat{c}')(s^t)$$

where the transformation of consumption into consumption shares is given by (4).

Detrended Arrow Economy We proceed as before, by conjecturing that the equilibrium consumption shares only depend on y^t. Our first result states that if the consumption shares in the de-trended economy do not depend on the aggregate history z^t, then it follows that the interest rates in this economy are deterministic.

Proposition A.2. *In the de-trended Arrow economy, if there exists a competitive equilibrium with equilibrium consumption allocations* $\left\{\hat{c}_t(\theta_0, y^t)\right\}$ *, then there is a deterministic interest rate process* $\left\{\hat{R}_t^A\right\}$ *and equilibrium prices* $\left\{\hat{q}_t(z^t, z_{t+1})\right\}$ *, that satisfy:*

$$\hat{q}_t(z^t, z_{t+1}) = \frac{\hat{\phi}(z_{t+1})}{\hat{R}_t^A} \tag{37}$$

All the results basically go through. We can map an equilibrium of the Bewley economy into an equilibrium of the detrended Arrow economy.

Theorem A.1. *An equilibrium of the Bewley model* $\{\hat{c}_t(\theta_0, y^t), \hat{a}_t(\theta_0, y^t), \hat{\sigma}_t(\theta_0, y^t)\}$ *and* $\{\hat{R}_t, \hat{v}_t\}$ *can be made into an equilibrium for the Arrow economy with growth,* $\left\{a_t(\theta_0, s^t, z_{t+1})\right\}$ *,* $\{\sigma_t(\theta_0, s^t)\}$*,* $\left\{c_t(\theta_0, s^t)\right\}$ *and* $\left\{q_t(z^t, z_{t+1})\right\}$ *,* $\{v_t(z^t)\}$*, with*

$$\begin{aligned}
c_t(\theta_0, s^t) &= \hat{c}_t(\theta_0, y^t) e_t(z^t) \\
\sigma_t(\theta_0, s^t) &= \hat{\sigma}_t(\theta_0, y^t) \\
a_t(\theta_0, s^t, z_{t+1}) &= \hat{a}_t(\theta_0, y^t) e_{t+1}(z^{t+1}) \\
v_t(z^t) &= \hat{v}_t e_t(z^t) \\
q_t(z^t, z_{t+1}) &= \frac{1}{\hat{R}_t} * \frac{\phi(z_{t+1})\lambda(z_{t+1})^{-\alpha}}{\sum_{z_{t+1}} \phi(z_{t+1})\lambda(z_{t+1})^{1-\alpha}}
\end{aligned}$$

As a result, even for an economy with agents who have these Epstein-Zin preferences, the risk premium is not affected[29].

B Proofs

Proof of Proposition 2.1:

Proof. First, we show that households rank consumption streams $\{c_t(s^t)\}$ in the original economy in exactly the same way as they rank growth-deflated consumption streams

$$\hat{c}_t(s^t) = \frac{c_t(s^t)}{e_t(z^t)}.$$

[29]However, the extension to non-i.i.d. aggregate consumption growth is non-trivial, because the taste shocks affect continuation utilities, and simply adjusting the interest rate may not be sufficient.

where $s^t = (z^t, y^t)$. Denote $U(c)(s^t)$ as continuation utility of an agent from consumption stream c, starting at history s^t. This continuation utility follows the simple recursion

$$U(c)(s^t) = u(c_t(s^t)) + \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) U(c)(s^t, s_{t+1}),$$

where it is understood that $(s^t, s_{t+1}) = (z^t, z_{t+1}, y^t, y_{t+1})$. Divide both sides by $e_t(s^t)^{1-\gamma}$ to obtain

$$\frac{U(c)(s^t)}{e_t(z^t)^{1-\gamma}} = u(\hat{c}_t(s^t)) + \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) \frac{e_{t+1}(z^{t+1})^{1-\gamma}}{e_t(z^t)^{1-\gamma}} \frac{U(c)(s^t, s_{t+1})}{e_{t+1}(z^{t+1})^{1-\gamma}}.$$

Define a new utility index $\hat{U}(\cdot)$ as follows:

$$\hat{U}(\hat{c})(s^t) = \frac{U(c)(s^t)}{e_t(z^t)^{1-\gamma}},$$

it follows that

$$\begin{aligned}
\hat{U}(\hat{c})(s^t) &= u(\hat{c}_t(s^t)) + \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) \lambda(z_{t+1})^{1-\gamma} \hat{U}(\hat{c})(s^t, s_{t+1}) \\
&= u(\hat{c}_t(s^t)) + \hat{\beta}(s_t) \sum_{s_{t+1}} \hat{\pi}(s_{t+1}|s_t) \hat{U}(\hat{c})(s^t, s_{t+1})
\end{aligned}$$

Thus it follows, for two consumption streams c and c', that

$$U(c)(s^t) \geq U(c')(s^t) \text{ if and only if } \hat{U}(\hat{c})(s^t) \geq \hat{U}(\hat{c}')(s^t)$$

i.e. the household orders original and growth-deflated consumption streams in exactly the same way. ■

Proof of Proposition 4.1:

Proof. First, we suppose the borrowing constraints are not binding, which is the easiest case. Assume the equilibrium allocations only depend on y^t, not on z^t. Then conditions 2.2 and 2.3 imply that the Euler equations of the Arrow economy, for the contingent claim and the stock respectively, read as follows:

$$1 = \frac{\hat{\beta}\hat{\phi}(z_{t+1})}{\hat{q}_t(z^t, z_{t+1})} \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \ \forall z_{t+1} \tag{38}$$

$$1 = \hat{\beta} \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)} \right] * \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))}. \tag{39}$$

In the first Euler equation, the only part that depends on z_{t+1} is $\frac{\hat{\phi}(z_{t+1})}{\hat{q}_t(z^t, z_{t+1})}$ which therefore implies that $\frac{\hat{\phi}(z_{t+1})}{\hat{q}_t(z^t, z_{t+1})}$ cannot depend on z_{t+1} : $\hat{q}_t(z^t, z_{t+1})$ is proportional to $\hat{\phi}(z_{t+1})$. Thus define $\hat{R}_t^A(z^t)$ by

$$\hat{q}_t(z^t, z_{t+1}) = \frac{\hat{\phi}(z_{t+1})}{\hat{R}_t^A(z^t)} \tag{40}$$

as the risk-free interest rate in the stationary Arrow economy. Using this condition, the Euler equation in (38) simplifies to the following expression:

$$1 = \hat{\beta}\hat{R}_t^A(z^t) \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \tag{41}$$

First, notice that apart from $\hat{R}_t^A(z^t)$ noting in this condition depends on z^t, so we can choose $\hat{R}_t^A(z^t) = \hat{R}_t^A$. ■

Proof of Corollary 4.1 :

Proof. The prices for Arrow securities in the original economy are given by

$$q_t(z^t, z_{t+1}) = \frac{\hat{q}_t(z^t, z_{t+1})}{\lambda(z_{t+1})} = \frac{\hat{\phi}(z_{t+1})}{\hat{R}_t^A \lambda(z_{t+1})}$$

and using the definition of the growth-adjusted transition probability $\hat{\phi}(z_{t+1})$:

$$q_t(z^t, z_{t+1}) = \frac{1}{\hat{R}_t^A} * \frac{\phi(z_{t+1}|z_t)\lambda(z_{t+1})^{-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{1-\gamma}} \tag{42}$$

Note that $q_t(z^t, z_{t+1})$ is proportional to $\phi(z_{t+1}|z_t)\lambda(z_{t+1})^{-\gamma}$, but not to $\phi(z_{t+1}|z_t)$.

If we denote by $R_t^A = \frac{1}{\sum_{z_{t+1}} q_t(z^t, z_{t+1})}$ the implied risk-free interest rate in the original economy, we can easily back it out from the risk-free rate in the detrended Arrow economy:

$$R_t^A(z^t) = \hat{R}_t^A(z^t) * \frac{\sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{1-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{-\gamma}} \tag{43}$$

And, obviously, $R_t^A(z^t) = R_t^A$, the risk-free rate in the Arrow economy does not depend on time either. ■

Proof of Corollary 4.2:

Proof. Then (also using the conjecture above and the form of $\hat{q}_t(z^t, z_{t+1})$) consider the budget constraint in the detrended Arrow economy:

$$\begin{aligned}
& \hat{c}_t(y^t) + \sum_{z_{t+1}} \hat{a}_t(y^t, z^t, z_{t+1})\hat{q}_t(z^t, z_{t+1}) + \sigma_t(y^t)\hat{v}_t \\
= \; & \eta(y_t) + \hat{a}_{t-1}(y^{t-1}, z^t) + \sigma_{t-1}(y^{t-1})\left[\hat{v}_t + \alpha\right] \\
& \hat{c}_t(y^t) + \frac{\sum_{z_{t+1}} \hat{a}_t(y^t, z^t, z_{t+1})\hat{\phi}(z_{t+1})}{\hat{R}_t^A} + \sigma_t(y^t)\hat{v}_t \\
= \; & \eta(y_t) + \hat{a}_{t-1}(y^{t-1}, z^t) + \sigma_{t-1}(y^{t-1})\left[\hat{v}_t + \alpha\right]
\end{aligned}$$

Note that only the contingent claims quantities themselves depend on aggregate histories. So, it is obvious we can simply choose $\hat{a}_t(s^t, z_{t+1}) = \frac{a_t(s^t, z_{t+1})}{e_{t+1}(z^{t+1})} = \hat{a}_t(y^t)$,that is, holdings of Arrow securities in the transformed economy also only depend on the idiosyncratic shock history. ■

Proof of Theorem 4.1:

Proof. The crucial observation is that at the prices $\left\{q_t(z^t, z_{t+1})\right\}$, $\{v_t(z^t)\}$ the consumption allocation $\left\{c_t(\theta_0, s^t)\right\}$ satisfies the Euler equations of the original problem, because

$$\begin{aligned}
& \frac{\hat{\beta}(s_t)}{\hat{q}_t(z^t, z_{t+1})} \sum_{y_{t+1}} \hat{\pi}(s_{t+1}|s_t) \frac{u'(\hat{c}_{t+1}(s^t, s_{t+1}))}{u'(\hat{c}_t(s^t))} \\
= \; & \frac{\beta \sum_{y_{t+1}} \pi(s_{t+1}|s_t) \left(\frac{e_{t+1}(z^{t+1})}{e_t(z^t)}\right)^{1-\gamma}}{q_t(z^t, z_{t+1}) \frac{e_{t+1}(z^{t+1})}{e_t(z^t)}} \left(\frac{c_{t+1}(s^t, s_{t+1})}{c_t(s^t)}\right)^{-\gamma} \left(\frac{e_{t+1}(z^{t+1})}{e_t(z^t)}\right)^{\gamma} \\
= \; & \frac{\beta}{q_t(z^t, z_{t+1})} \sum_{y_{t+1}} \pi(s_{t+1}|s_t) \left(\frac{c_{t+1}(s^t, s_{t+1})}{c_t(s^t)}\right)^{-\gamma}
\end{aligned}$$

and

$$
\begin{aligned}
& \hat{\beta}(s_t) \sum_{s_{t+1}} \hat{\pi}(s_{t+1}|s_t) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)}\right] \frac{u'(\hat{c}_{t+1}(s^t, s_{t+1}))}{u'(\hat{c}_t(s^t))} \\
= & \ \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) \left(\frac{e_{t+1}(z^{t+1})}{e_t(z^t)}\right)^{1-\gamma} \left[\frac{v_{t+1}(z^{t+1}) + \alpha e_{t+1}(z^{t+1})}{v_t(z^t)}\right] \left(\frac{e_{t+1}(z^{t+1})}{e_t(z^t)}\right)^{-1} \\
& * \left(\frac{c_{t+1}(s^t, s_{t+1})}{c_t(s^t)}\right)^{-\gamma} \left(\frac{e_{t+1}(z^{t+1})}{e_t(z^t)}\right)^{\gamma} \\
= & \ \beta \sum_{s_{t+1}} \pi(s_{t+1}|s_t) \left[\frac{v_{t+1}(z^{t+1}) + \alpha e_{t+1}(z^{t+1})}{v_t(z^t)}\right] \left(\frac{c_{t+1}(s^t, s_{t+1})}{c_t(s^t)}\right)^{-\gamma},
\end{aligned}
$$

which shows that the non-detrended variables satisfy the Euler equations of the non-detrended economy without borrowing constraints. Especially it shows that it was suitable to define $\hat{q}_t(z^t, z_{t+1})$ and $\hat{v}_t(z^t)$ in the way we have above. ■

Proof of Corollary 4.4:

Proof. Suppose the stationary Bewley allocation $\{\hat{a}_t(y^t), \hat{\sigma}_t(y^t)\}$ satisfies the constraint

$$\frac{\hat{a}_t(y^t)}{\hat{R}_t} + \hat{\sigma}_t(y^t)\hat{v}_t \geq \hat{K}_t(y^t)$$

which seems the natural constraint to impose on the stationary Bewley economy. We want to show that the allocation for the stochastic Arrow economy satisfies the borrowing constraint if the allocation for the stationary Bewley economy does. Multiply both sides by $e_t(z^t)\hat{\phi}(z_{t+1})$ to obtain

$$\frac{a_t(s^t, z_{t+1})\hat{\phi}(z_{t+1})}{\hat{R}_t\lambda(z_{t+1})} + \sigma_t(s^t)v_t(z^t)\hat{\phi}(z_{t+1}) \geq K_t(s^t)\hat{\phi}(z_{t+1})$$

Using the fact that

$$q_t(z^t, z_{t+1}) = \frac{\hat{\phi}(z_{t+1})}{\hat{R}_t\lambda(z_{t+1})}$$

and summing over all z_{t+1} yields

$$\sum_{z_{t+1}} q_t(z^t, z_{t+1})a_t(s^t, z_{t+1}) + \sigma_t(s^t)v_t(z^t) \geq K_t(s^t),$$

exactly the constraint of the stochastic Arrow economy. So for this borrowing constraint and the Arrow economy the above Theorem goes through unchanged. Next consider the state-by state wealth constraint. In the stationary Bewley economy we may impose

$$\hat{a}_t(y^t) + \hat{\sigma}_t(y^t)\left[\hat{v}_{t+1} + \alpha\right] \geq \hat{M}_t(y^t)$$

Multiplying by $e_{t+1}(z^{t+1})$ yields

$$a_t(s^t, z_{t+1}) + \sigma_t(s^t)\left[v_{t+1}(z^{t+1}) + \alpha e_{t+1}(z^{t+1})\right] \geq M_t(s^t, z_{t+1}) \text{ for all } z_{t+1}$$

and thus also for this constraint the Theorem above goes through, because, if the borrowing constraints are binding, we now know that the detrended version of the Arrow economy and the Bewley economy have the same Euler equations, even if the constraints bind: the Lagrangian multipliers $\mu_t(y^t)$ and $\kappa_t(y^t)$ from the Bewley economy, together with the allocations $\left\{\hat{c}_t(\theta_0, y^t)\right\}$, are also the right multipliers for the detrended Arrow economy. Proposition (4.1) is still valid in the case of binding constraints. ■

Proof of Proposition 5.1:

Proof. Again suppose the borrowing constraints are not binding. Conditions 2.2 and 2.3, and our conjecture that the allocations only depend on y^t imply that the Euler equations of the *HL* economy read as

$$1 = \hat{\beta} R_t(z^t) \sum_{z_{t+1}} \frac{\hat{\phi}(z_{t+1})}{\lambda(z_{t+1})} * \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \tag{44}$$

$$1 = \hat{\beta} \sum_{s_{t+1}} \hat{\phi}(z_{t+1}) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)}\right] * \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \tag{45}$$

Define

$$\hat{R}_t^{IC}(z^t) = R_t(z^t) * \frac{\sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{1-\gamma}} \tag{46}$$

as the risk-free interest rate in the transformed *HL* economy. It is immediate that equation (44) becomes

$$1 = \hat{\beta} \hat{R}_t^{HL}(z^t) \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))} \tag{47}$$

which is identical to the Euler equation (41) in the Arrow economy. This suggests (since the

Euler equations are identical in both economies) that we have

$$\hat{R}_t^{HL}(z^t) = \hat{R}_t^{A}(z^t) \tag{48}$$

Notice that since

$$R_t(z^t) \equiv R_t^{HL}(z^t) = \hat{R}_t^{HL}(z^t) * \frac{\sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{1-\gamma}}{\sum_{z_{t+1}} \phi(z_{t+1}|z_t)\lambda(z_{t+1})^{-\gamma}} \tag{49}$$

it then follows that

$$R_t^{HL}(z^t) = R_t^{A}(z^t) \tag{50}$$

that is, the implied risk-free rates in the original Arrow and *HL* economies are identical. ∎

Proof of Lemma 5.1

Proof. Recall that the budget constraint for the transformed *HL* economy is

$$\hat{c}_t(s^t) + \frac{\hat{b}_t(s^t)}{R_t(z^t)} + \sigma_t(s^t)\hat{v}_t(z^t) \leq \eta(y_t) + \frac{\hat{b}_{t-1}(s^{t-1})}{\lambda(z_t)} + \sigma_{t-1}(s^{t-1})\left[\hat{v}_t(z^t) + \alpha\right]$$

Using the results from above

$$\hat{c}_t(y^t) + \frac{\hat{b}_t(s^t)}{R_t} + \sigma_t(y^t)\hat{v}_t \leq \eta(y_t) + \frac{\hat{b}_{t-1}(s^{t-1})}{\lambda(z_t)} + \sigma_{t-1}(y^{t-1})\left[\hat{v}_t + \alpha\right]$$

The budget constraint reads as

$$\hat{c}_t(y^t) - \eta(y_t) + \frac{\hat{b}_t(s^t)}{R_t} + \sigma_t(s^t)\hat{v}_t = \frac{\hat{b}_{t-1}(s^{t-1})}{\lambda(z_t)} + \sigma_{t-1}(s^{t-1})\left[\hat{v}_t + \alpha\right]$$

and from the absence of arbitrage

$$\frac{R_t}{\lambda_{t+1}} = \frac{\hat{v}_{t+1} + \alpha}{\hat{v}_t}$$

Next period's budget constraint reads as

$$\hat{c}_{t+1}(y^{t+1}) - \eta(y_{t+1}) + \frac{\hat{b}_{t+1}(s^{t+1})}{R_{t+1}} + \sigma_{t+1}(s^{t+1})\hat{v}_{t+1} = \frac{\hat{b}_t(s^t)}{\lambda(z_{t+1})} + \sigma_t(s^t)\left[\hat{v}_{t+1} + \alpha\right]$$

Multiplying both sides by $\hat{\phi}(z_{t+1})$ and then summing with respect to z_{t+1} yields

$$
\begin{aligned}
& \hat{c}_{t+1}(y^{t+1}) - \eta(y_{t+1}) + \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) \left(\frac{\hat{b}_{t+1}(s^{t+1})}{R_{t+1}} + \sigma_{t+1}(s^{t+1})\hat{v}_{t+1} \right) \\
= \quad & \hat{b}_t(s^t) \sum_{z_{t+1}} \frac{\hat{\phi}(z_{t+1})}{\lambda(z_{t+1})} + \sigma_t(s^t)\left[\hat{v}_{t+1} + \alpha\right] \\
= \quad & \hat{R}_t \left[\frac{\hat{b}_t(s^t)}{R_t} + \sigma_t(s^t)\hat{v}_t \right]
\end{aligned}
$$

(the last step requires the use of the definition of $\hat{R}_t$ and a little algebra). Thus

$$
\frac{\hat{b}_t(s^t)}{R_t} + \sigma_t(s^t)\hat{v}_t = \frac{\hat{c}_{t+1}(y^{t+1}) - \eta(y_{t+1})}{\hat{R}_t} + \frac{1}{\hat{R}_t} \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) \left(\frac{\hat{b}_{t+1}(s^{t+1})}{R_{t+1}} + \sigma_{t+1}(s^{t+1})\hat{v}_{t+1} \right)
$$

Repeating this and noting that by assumption future $\hat{c}$'s do not depend on $z_{t+\tau}$'s suggests

$$
\hat{c}_t(y^t) - \eta(y_t) + \sum_{\tau=t+1}^{\infty} \frac{\hat{c}_\tau(y^\tau) - \eta(y_\tau)}{\prod_{j=t}^{\tau-1} \hat{R}_j} = \frac{\hat{b}_{t-1}(s^{t-1})}{\lambda(z_t)} + \sigma_{t-1}(s^{t-1})\left[\hat{v}_t + \alpha\right]
$$

which simply says that if consumption in the detrended bond economy only depends on idiosyncratic shocks, then the wealth households come into the period with can only depend on this as well, that is,

$$
\frac{\hat{b}_{t-1}(s^{t-1})}{\lambda(z_t)} + \sigma_{t-1}(s^{t-1})\left[\hat{v}_t + \alpha\right]
$$

cannot depend on z_t. ■

Proof of Proposition 5.2:

Proof. Let us start with the case of 2 aggregate states. We consider the agent's wealth at the start of period t in the *IC* economy, for the case with two aggregate states:

$$
b_{t-1}(s^{t-1}) + \sigma_{t-1}(s^{t-1})\left[v_t(z^t) + \alpha e_t(z^t)\right] = \theta_{t-1}(s^{t-1}, hi),
$$

and, similarly, in the low state:

$$
b_{t-1}(s^{t-1}) + \sigma_{t-1}(s^{t-1})\left[v_t(z^t) + \alpha e_t(z^t)\right] e_{t-1}(z^{t-1}) = \theta_{t-1}(s^{t-1}, lo),
$$

where the $\theta_{t-1}(s^{t-1}, z_t)$ are the contingent claim positions (total, including stocks) in the Arrow economy *at the start of t*. We deflate the wealth positions of our agent in each aggregate state, to obtain:

$$\hat{b}_{t-1}(s^{t-1}) + \sigma_{t-1}^{HL}(s^{t-1}) \left[\hat{v}_t(z^t) + \alpha\right] \lambda(hi) = \hat{\theta}_{t-1}(s^{t-1}, hi)\lambda(hi)$$

and, similarly, in the low state:

$$\hat{b}_{t-1}(s^{t-1}) + \sigma_{t-1}^{HL}(s^{t-1}) \left[\hat{v}_t(z^t) + \alpha\right] \lambda(lo) = \hat{\theta}_{t-1}(s^{t-1}, lo)\lambda(lo)$$

This implies that the stock position of the agent is given by:

$$\sigma_{t-1}^{HL}(s^{t-1}) \left[\hat{v}_t(z^t) + \alpha\right] = \frac{\left[\hat{\theta}_{t-1}(s^{t-1}, hi)\lambda(hi) - \hat{\theta}_{t-1}(s^{t-1}, lo)\lambda(lo)\right]}{\left[\lambda(hi) - \lambda(lo)\right]}$$

If we impose that $\hat{\theta}_{t-1}(s^{t-1}, lo) = \hat{\theta}_{t-1}(s^{t-1}, hi) = \hat{\theta}_{t-1}(s^{t-1})$, we simply obtain the following expression for the stock position:

$$\sigma_{t-1}^{IC}(s^{t-1}) = \frac{\hat{\theta}_{t-1}(s^{t-1})}{\left[\hat{v}_t + \alpha\right]}$$

where I have assumed that $\hat{v}_t(z^t) = \hat{v}_t$, while the bond position is given by;

$$\hat{b}_{t-1}(s^{t-1}) + \frac{\hat{\theta}_{t-1}(s^{t-1})}{\left[\hat{v}_t(z^t) + \alpha\right]} \left[\hat{v}_t(z^t) + \alpha\right] \lambda(z_t) = \hat{\theta}_{t-1}(s^{t-1})\lambda(z_t),$$

which implies the bond position is $\hat{b}_{t-1}(s^{t-1}) = 0$ for all s^{t-1}. So all of the state-contingency is obtained by going long or short in the stock, depending on whether you're borrowing or lending in the deflated economy. In the two-state case, the bond is redundant, it seems. The only way to get some trade in bonds is by imposing a short-sales constraint on the stock. This argument generalizes to the case of N aggregate states quite easily. We just set the stock position equal to:

$$\sigma_{t-1}^{HL}(s^{t-1}) = \frac{\hat{\theta}_{t-1}(s^{t-1})}{\left[\hat{v}_t + \alpha\right]}$$

and we simply let $\hat{b}_{t-1}(s^{t-1}) = 0$. This immediately implies that the wealth position at the start of t in the HL economy coincides with that in the Arrow economy for all z_t :

$$b_{t-1}(s^{t-1}) + \sigma_{t-1}^{HL}(s^{t-1}) \left[v_t(z^t) + \alpha e_t(z^t)\right] = \theta_{t-1}^{A}(s^{t-1}, z_t),$$

■

Proof of Corollary 5.1

Proof. We constructed the same net wealth positions at the start of each period in the *HL* economy as in the Arrow economy. As a result, the solvency constraints imposed in the Arrow economy will be satisfied in the *HL* economy. In the stationary Arrow economy the constraints were

$$\begin{aligned}\frac{\hat{a}_t(y^t)}{\hat{R}_t} + \hat{\sigma}_t^A(y^t)\hat{v}_t &\geq \hat{K}_t(y^t) \\ \hat{a}_t(y^t) + \hat{\sigma}_t(y^t)\left[\hat{v}_{t+1} + \alpha\right] &\geq \hat{M}_t(y^t)\end{aligned}$$

But from (**??**) we have

$$\begin{aligned}\frac{\hat{b}_t(s^t)}{R_t(z^t)} + \hat{\sigma}_t^{IC}(y^t)\hat{v}_t &= \frac{\hat{a}_t(y^t)}{\hat{R}_t} + \hat{\sigma}_t^A(y^t)\hat{v}_t \geq \hat{K}_t(y^t) \\ \frac{\hat{b}_t(s^t)}{\lambda(z_{t+1})} + \sigma_t^{IC}(y^t)\left[\hat{v}_{t+1} + \alpha\right] &= \hat{a}_t(y^t) + \hat{\sigma}_t^A(y^t)\left[\hat{v}_{t+1} + \alpha\right] \geq \hat{M}_t(y^t)\end{aligned}$$

and thus we know that the asset position for the bond economy satisfy the shortsale constraint, which bind if and only if they bind in the stationary Arrow (or Bewley) economy. So we can indeed use the same Lagrange multipliers. ■

Proof of :A.1

Proof. The first order conditions for the representative agent are given by:

$$\begin{aligned}1 &= \frac{\hat{\beta}\hat{\phi}(z_{t+1})}{\hat{q}_t(z^t, z_{t+1})} \frac{u'(\hat{c}_{t+1}(z^t, z_{t+1}))}{u'(\hat{c}_t(z^t))} \; \forall z_{t+1} \\ 1 &= \hat{\beta} \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)}\right] \frac{u'(\hat{c}_{t+1}(z^t, z_{t+1}))}{u'(\hat{c}_t(z^t))} \qquad (51)\end{aligned}$$

■

Proof of Lemma 6.1:

Proof. First, consider a household whose borrowing constraints do not bind. The Euler equation of the Arrow/*HL* economy for the stock respectively reads as follows:

$$1 = \hat{\beta} \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)} \right] * \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))}.$$

The Euler equation in the Arrow/*HL* economy for the bond reads as follows:

$$1 = \hat{\beta} \hat{R}_t \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))}$$

This implies that

$$\hat{\beta} \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)} - \hat{R}_t \right] = 0$$

which then also holds if we replace $\hat{\beta}$ by β. This immediately implies that

$$E_t[\left(R^s_{t+1} - R_t\right) \beta \left(\frac{e_{t+1}}{e_t} \right)^{-\gamma}] = 0.$$

In the Arrow economy, the Consumption-CAPM prices any excess return $R^i_{t+1} - R_t$ as long as the returns only depend on the aggregate state z_{t+1}. In the *HL* model, it only prices a claim to aggregate consumption.

■

Proof of Proposition 6.1:

Proof. Let us define

$$1/\hat{R}^{RE}_\tau = 1/\hat{R}^{RE} = \sum_{z_{t+1}} \hat{q}_t(z^t, z_{t+1}) = \sum_{z_{t+1}} \hat{\beta} \hat{\phi}(z_{t+1}) = \hat{\beta}.$$

We know that in the Arrow economy, the equilibrium Arrow prices are given by:

$$q^A_t(z^t, z_{t+1}) = \frac{\hat{q}^A_t(z^t, z_{t+1})}{\lambda(z_{t+1})} = \frac{\hat{\phi}(z_{t+1})}{\lambda(z_{t+1}) \hat{R}^A_t(z^t)}$$

whereas in the representative agent economy Arrow prices are given as

$$q_t(z^t, z_{t+1}) = \frac{\hat{q}_t(z^t, z_{t+1})}{\lambda(z_{t+1})} = \hat{\beta} \frac{\hat{\phi}(z_{t+1})}{\lambda(z_{t+1})}$$

This implies that the stochastic discount factor in the Arrow economy equals the SDF in the representative agent economy multiplied by a non-random number κ_t :

$$m^A_{t+1} = m^{RE}_{t+1}\kappa_t$$

with $\kappa_t = \frac{\hat{R}^E_t}{\hat{R}^A_t}$ and $\left(\hat{R}^E\right)^{-1} = \hat{\beta}$. ■

Proof of Theorem 6.1:

Proof. First, note that the multiplicative risk premium can be stated as a weighted sum of risk premia on strips ((Alvarez and Jermann 2001)):

$$\begin{aligned} 1+\nu_t &= E_t m_{t+1} E_t \left(\frac{\sum_{k=1}^{\infty} E_{t+1} m_{t+1,t+k} \alpha e_{t+k}}{\sum_{k=1}^{\infty} E_t m_{t,t+k} \alpha e_{t+k}} \right) \\ &= \sum_{k=1}^{\infty} \frac{\frac{E_{t+1} m_{t,t+k} e_{t+k}}{E_t m_{t,t+k} e_{t+k}}}{1/E_t m_{t+1}} \frac{E_t m_{t,t+k} e_{t+k}}{\sum_{k=1}^{\infty} E_t m_{t,t+k} e_{t+k}} \\ &= \sum_{k=1}^{\infty} \omega_k \frac{E_t R_{t,1}\left[e_{t+k}\right]}{R_{t,1}\left[1\right]}, \end{aligned}$$

where the weights are

$$\omega_k = \frac{E_t m_{t,t+k} e_{t+k}(z^{t+k})}{\sum_{l=1}^{\infty} E_t m_{t,t+l} e_{t+l}(z^{t+k})} \tag{52}$$

The multiplicative risk premium on a one-period strip (a claim to the Lucas tree's dividend next period only, not the entire stream) is the same in the Arrow economy as in the representative agent economy. First, we show that the one-period ahead conditional strip risk premia are identical:

$$\frac{E_t \frac{\alpha e_{t+1}(z^{t+1})}{E_t[m^A_{t+1}\alpha e_{t+1}(z^{t+1})]}}{\frac{1}{E_t[m^A_{t+1}]}} = \frac{E_t \frac{\lambda_{t+1}(z^{t+1})}{E_t[m^A_{t+1}\lambda_{t+1}(z^{t+1})]}}{\frac{1}{E_t[m^A_{t+1}]}} = \frac{E_t \frac{\lambda_{t+1}(z^{t+1})}{E_t[m^{RE}_{t+1}\lambda_{t+1}(z^{t+1})]}}{\frac{1}{E_t[m^{RE}_{t+1}]}}$$

Next, we follow Alvarez and Jermann's proof strategy (in a different setting), and we show that the risk premia on k-period strips are identical. Here is the risk premium on a $k-$period strip:

$$\frac{E_t R_{t,t+1}\left[\alpha e_{t+k}\right]}{E_t R_{t,t+1}\left[1\right]} = \frac{E_t \frac{E_{t+1}[m^A_{t+1,t+k}\alpha e_{t+k}(z^{t+k})]}{E_t[m^A_{t,t+k}\alpha e_{t+k}(z^{t+k})]}}{\frac{1}{E_t[m^A_{t+1}]}}$$

Now, since the aggregate shocks are iid, the term structure of the risk premia in the representative agent economy is flat (i.e. the risk premia does not depend on k) :

$$\frac{E_t R^{RE}_{t,t+1}\left[\alpha e_{t+k}\right]}{E_t R^{RE}_{t,t+1}\left[1\right]} = \frac{E_t \frac{[E\lambda(z)^{1-\gamma}]^{k-1}}{[E\lambda(z)^{1-\gamma}]^{k}}}{\frac{1}{E_t[m^{RE}_{t+1}]}} = \frac{\frac{1}{E[\lambda(z)^{1-\gamma}]}}{\frac{1}{E[\lambda(z)^{-\gamma}]}} = \frac{E[\lambda(z)^{-\gamma}]}{E[\lambda(z)^{1-\gamma}]}$$

(same proof by Alvarez and Jermann on page 37.) Now, to keep things simple, assume the $R^A_t = R^{RE}$ for all $t, and\ \kappa_t$ is constant as a result, then this implies that then the term structure of risk premia in the Arrow economy is flat as well.

$$\frac{E_t R^{RE}_{t,t+1}\left[\alpha e_{t+k}\right]}{E_t R^{RE}_{t,t+1}\left[1\right]} = \frac{E_t \frac{[E\lambda(z)^{1-\gamma}\kappa]^{k-1}}{[E\lambda(z)^{1-\gamma}\kappa]^{k}}}{\frac{1}{E_t[m^{rep}_{t+1}]}} = \frac{\frac{1}{E[\lambda(z)^{1-\gamma}\kappa]}}{\frac{1}{E[\lambda(z)^{-\gamma}\kappa]}} = \frac{E[\lambda(z)^{-\gamma}]}{E[\lambda(z)^{1-\gamma}]}$$

But this implies that the multiplicative risk premia is unchanged since the risk premia on the consumption strips are invariant in k and the one-period ahead .This means the multiplicative risk premium is unchanged. The proof goes through for time-varying κ_t, but the algebra is a little messier. ∎

Proof of Lemma 7.1:

Proof. By construction, the Euler equations of the Arrow model with these prices and the consumption allocations from the Bewley model are satisfied. We start by assuming the borrowing constraint does not bind. First, note that the Euler equation in the Bewley economy is given by:

$$1 = \frac{\tilde{\beta}_{t+1}}{\tilde{\beta}_t}\hat{R}_t \sum_{y_{t+1}} \varphi(y_{t+1}|y_t)\frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))}.$$

The Euler equation in the detrended Arrow economy implies that:

$$\sum_{z_{t+1}} \hat{q}_t(z^t, z_{t+1}) = \hat{\beta}(z_t)\sum_{y_{t+1}} \varphi(y_{t+1}|y_t)\frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))}\forall z_{t+1},$$

which in turn implies that:

$$\frac{1}{\hat{R}_t}\frac{\widetilde{\beta}_t}{\widetilde{\beta}_{t+1}} = \sum_{y_{t+1}} \varphi(y_{t+1}|y_t)\frac{u'(\hat{c}_{t+1}(y^t, y_{t+1}))}{u'(\hat{c}_t(y^t))}\forall z_{t+1}$$

This is exactly the same Euler equation as in the Bewley economy. ∎

The state-contingent interest rate in this economy is given by:

$$\frac{1}{\hat{R}_t^A(z_t)} = \hat{\beta}(z_t)\frac{\widetilde{\beta}_t}{\hat{R}_t\widetilde{\beta}_{t+1}}$$

which can easily be verified from equation (25).

Proof of Proposition 7.1:

Proof. We need to check that these allocations satisfy the budget constraint in the economy with time-varying discount factors. Let us take $\{\hat{c}_t(\theta_0, y^t), \hat{a}_t(\theta_0, y^t) = 0, \hat{\sigma}_t(\theta_0, y^t)\}$ from the Bewley model and let us back out the implied state-contingent Arrow securities positions recursively, for each y^t, z_t, from the budget constraint of the Arrow model

$$\begin{aligned} \hat{a}_{t-1}(y^{t-1}, z^t, \theta_0) &= \hat{c}_t(y^t, \theta_0) - \eta(y_t) + \sum_{\tau=t+1}^{\infty}\sum_{z^\tau, y^\tau} \hat{Q}_t(z^\tau|z_t)\left(\hat{c}_\tau(y^\tau, \theta_0) - \eta(y_\tau)\right) \\ &\quad -\sigma_{t-1}(y^{t-1})\left[\hat{v}_t(z_t) + \alpha\right] = \hat{a}_{t-1}(y^{t-1}, z_t, \theta_0) \end{aligned} \tag{53}$$

This ensures that $\left\{\hat{a}_{t-1}(y^{t-1}, z_t, \theta_0), \sigma_{t-1}(y^{t-1}, \theta_0)\right\}$ finance the consumption allocations $\{\hat{c}_t(\theta_0, y^t)\}$. Next, we need to check that the market for each Arrow security clears. That is, we have to check that

$$\begin{aligned} \int \sum_{y^{t-1}} \pi(y^{t-1}|y_0)\hat{a}_{t-1}(y^{t-1}, z_t, \theta_0)d\Theta_0 &= \\ \sum_{y^t|y^{t-1}} \pi(y^t|y^{t-1}) \int \sum_{y^{t-1}} \pi(y^{t-1}|y_0)\hat{a}_{t-1}(y^{t-1}, z_t, \theta_0)d\Theta_0 &= \\ \int \sum_{y^t} \pi(y^t|y_0)\hat{a}_{t-1}(y^{t-1}, z_t, \theta_0)d\Theta_0 &= 0 \end{aligned}$$

for each z_t. We now multiply equation (53) by $\pi(y^t|y_0)$ and then sum over all y^t. For the first term we obtain, using the aggregate resource constraint,

$$\begin{aligned} &\int \sum_{y^{t-1}} \pi(y^{t-1}|y_0) \sum_{y^t} \pi(y^t|y_{t-1})\left(\hat{c}_t(y^t, \theta_0) - \eta(y_t)\right) d\Theta_0 \\ = &\int \sum_{y^t} \pi(y^t|y_0)\left(\hat{c}_t(y^t, \theta_0) - \eta(y_t)\right) d\Theta_0 = \alpha. \end{aligned} \tag{54}$$

By the same token we obtain

$$
\begin{aligned}
&\int \sum_{y^{t-1}} \pi(y^{t-1}|y_0) \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \hat{Q}_t(z^\tau|z_t) \left(\hat{c}_\tau(y^\tau, \theta_0) - \eta(y_\tau)\right) d\Theta_0 \qquad (55)\\
= \quad &\sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \hat{Q}_t(z^\tau|z_t)\alpha.
\end{aligned}
$$

Furthermore we know from the market clearing in the market for shares that

$$
\int \sum_{y^{t-1}} \pi(y^{t-1}|y_0)\sigma_{t-1}(y^{t-1}, \theta_0) \left[\hat{v}_t(z_t) + \alpha\right] d\Theta_0 = \left[\hat{v}_t(z_t) + \alpha\right], \qquad (56)
$$

and finally we know that the share price can be written as follows:

$$
\left[\hat{v}_t(z_t) + \alpha\right] = \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \hat{Q}_t(z^\tau|z_t)\alpha + \alpha, \qquad (57)
$$

which in turn implies, substituting (54), (55) and (56) into (53) that

$$
\frac{1}{\lambda(z_t)} \int \sum_{y^{t-1}} \pi(y^{t-1}|y_0)\hat{a}_{t-1}(y^{t-1}, z_t) d\Theta_0 = 0
$$

Thus, each of the Arrow securities markets clears for the new trading strategies. ■

Proof of Lemma 7.2:

Proof. From the definition of $\left\{\widetilde{\beta}_t\right\}$, it follows that the Arrow-Debreu prices at time 0 before the realization of z_0:

$$
\begin{aligned}
\hat{Q}_{0,\tau} \quad &= \quad \sum_{z_0} \hat{\Pi}(z_0) \sum_{z^\tau} \hat{Q}_\tau(z^\tau|z_0) = \sum_{z_0} \hat{\Pi}(z_0) \sum_{z^{\tau-1}} \frac{\widehat{\phi}(z^\tau|z_0)\hat{\beta}_{0,\tau-1}(z^{\tau-1}|z_0)}{\widetilde{\beta}_\tau \widehat{R}_{0,\tau}}\\
&= \quad \frac{1}{\widetilde{\beta}_\tau^\tau \widehat{R}_{0,\tau}} \sum_{z_0} \hat{\Pi}(z_0) \sum_{z^\tau} \widehat{\phi}(z^\tau|z_0)\hat{\beta}_{0,\tau-1}(z^{\tau-1}|z_0)\\
&= \quad \frac{\widetilde{\beta}_\tau^\tau}{\widetilde{\beta}_\tau^\tau \widehat{R}_{0,\tau}} = \tilde{Q}_{0,\tau}
\end{aligned}
$$

where we have used the result in equation (24). ■

Proof of Corollary 7.3:

Proof.

$$\begin{aligned}
\sum_{z_0} \hat{\Pi}(z_0)\hat{a}_{-1}(z_0,\theta_0) &= \sum_{z_0} \hat{\Pi}(z_0) \sum_{\tau=1}^{\infty} \sum_{z^\tau} \left(\hat{Q}_\tau(z^\tau|z_0) - \widetilde{Q}_\tau \right) \sum_{y^\tau} \left(\hat{c}_\tau(y^\tau,\theta_0) - \eta(y_\tau) \right) \\
&\quad -\sigma_{-1} \sum_{z_0} \hat{\Pi}(z_0) \left[\hat{v}_t(z_t) - \widetilde{\nu}_t(z^t) \right]. \\
&= \sum_{z_0} \hat{\Pi}(z_0) \sum_{\tau=1}^{\infty} \sum_{z^\tau} \left(\hat{Q}_\tau(z^\tau|z_0) - \widetilde{Q}_\tau \right) \sum_{y^\tau} \left(\hat{c}_\tau(y^\tau,\theta_0) - \eta(y_\tau) \right) \\
&\quad -\sigma_{-1}\alpha \sum_{z_0} \hat{\Pi}(z_0) \sum_{\tau=1}^{\infty} \sum_{z^\tau} \left(\hat{Q}_\tau(z^\tau|z_0) - \widetilde{Q}_\tau \right) = 0,
\end{aligned}$$

because we know that:

$$\sum_{z_0} \hat{\Pi}(z_0) \sum_{z^t} \hat{Q}_t(z^t|z_0) \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \left(\hat{Q}_\tau(z^\tau|z_t) - \widetilde{Q}_\tau \right) = 0$$

■

Proof. of Lemma 7.4: Suppose the stationary Bewley allocation $\{\widetilde{\hat{a}}_t(y^t), \hat{\sigma}_t(y^t)\}$ satisfies the constraint

$$\frac{\widetilde{\hat{a}}_t(y^t)}{\hat{R}_t} + \hat{\sigma}_t(y^t)\hat{v}_t \geq \hat{K}_t(y^t)$$

which seems the natural constraint to impose on the stationary Bewley economy. We want to show that the allocation for the stochastic Arrow economy satisfies the borrowing constraint if the allocation for the stationary Bewley economy does. Multiply both sides by $e_t(z^t)\hat{\phi}(z_{t+1})$ to obtain

$$\frac{\widetilde{a}_t(s^t, z_{t+1})\hat{\phi}(z_{t+1})}{\hat{R}_t\lambda(z_{t+1})} + \sigma_t(s^t)v_t(z^t)\hat{\phi}(z_{t+1}) \geq K_t(s^t)\hat{\phi}(z_{t+1})$$

Using the fact that

$$q_t(z^t, z_{t+1}) = \frac{\hat{\phi}(z_{t+1})}{\hat{R}_t\lambda(z_{t+1})}$$

and summing over all z_{t+1} yields

$$\sum_{z_{t+1}} q_t(z^t, z_{t+1})\widetilde{a}_t(s^t, z_{t+1}) + \sigma_t(s^t)v_t(z^t) \geq K_t(s^t),$$

exactly the constraint of the stochastic Arrow economy, but the actual bond position differs from $\widetilde{a}_t(s^t, z_{t+1})$, because

$$\frac{\hat{a}_{t-1}(y^{t-1}, z_t, \theta_0)}{\lambda(z_t)} - \frac{\widetilde{a}_{t-1}(y^{t-1}, z_t, \theta_0)}{\lambda(z_t)} = \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \left(\hat{Q}_\tau(z^\tau | z_t) - \widetilde{Q}_\tau \right) \sum_{\eta^\tau} \left(\hat{c}_\tau(y^\tau, \theta_0) - \eta(y_\tau) \right)$$
$$-\sigma_{t-1}(y^{t-1}) \alpha \sum_{\tau=t+1}^{\infty} \sum_{z^\tau} \left(\hat{Q}_\tau(z^\tau | z_t) - \widetilde{Q}_\tau \right)$$

So, we know that

$$\sum_{z_{t+1}} q_t(z^t, z_{t+1}) a_t(s^t, z_{t+1}) + \sigma_t(s^t) v_t(z^t) \geq K_t(s^t) + \sum_{z_{t+1}} q_t(z^t, z_{t+1}) \left(\begin{array}{c} \widetilde{a}_t(s^t, z_{t+1}) \\ -a_t(s^t, z_{t+1}) \end{array} \right),$$

So, we can simply re-define the solvency constraints as :

$$K_t^*(s^t) = K_t(s^t) + \sum_{z_{t+1}} q_t(z^t, z_{t+1}) \left(\begin{array}{c} \widetilde{a}_t(s^t, z_{t+1}) \\ -a_t(s^t, z_{t+1}) \end{array} \right)$$

Next consider the state-by state wealth constraint. In the stationary Bewley economy we may impose

$$\hat{a}_t(y^t) + \hat{\sigma}_t(y^t) \left[\hat{v}_{t+1} + \alpha \right] \geq \hat{M}_t(y^t)$$

Multiplying by $e_{t+1}(z^{t+1})$ yields

$$\widetilde{a}_t(s^t, z_{t+1}) + \sigma_t(s^t) \left[v_{t+1}(z^{t+1}) + \alpha e_{t+1}(z^{t+1}) \right] \geq M_t(s^t, z_{t+1}) \text{ for all } z_{t+1}$$

By the same token, we can redefine:

$$M_{t+1}^*(s^{t+1}) = M_{t+1}(s^{t+1}) + \widetilde{a}_t(s^t, z_{t+1}) - a_t(s^t, z_{t+1})$$

■

Proof of Proposition A.1:

Proof. First, we divided through by $e_t(z^t)$ on both sides in equation (36):

$$\begin{aligned}
\frac{V_t(s^t)}{e_t(z^t)} &= \left[(1-\beta)\frac{c_t^{1-\rho}}{e_t^{1-\rho}} + \beta\frac{(\mathcal{R}_t V_{t+1})^{1-\rho}}{e_t^{1-\rho}}\right]^{\frac{1}{1-\rho}} \\
\hat{V}_t(s^t) &= \left[(1-\beta)\hat{c}_t^{1-\rho} + \beta(\frac{\mathcal{R}_t V_{t+1}}{e_t})^{1-\rho}\right]^{\frac{1}{1-\rho}}. \qquad (58)
\end{aligned}$$

Note that the risk-adjusted continuation utility can be stated as:

$$\begin{aligned}
\frac{\mathcal{R}_t V_{t+1}}{e_t(z^t)} &= \left(E_t\left(\frac{e_{t+1}}{e_t}\right)^{1-\alpha}\frac{V_{t+1}^{1-\alpha}}{e_{t+1}^{1-\alpha}}\right)^{1/1-\alpha} \\
&= \left(\sum_{s_{t+1}} \pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}\hat{V}_{t+1}^{1-\alpha}(s_{t+1})\right)^{1/1-\alpha}
\end{aligned}$$

Next, we define growth-adjusted probabilities and the growth-adjusted discount factor as:

$$\hat{\pi}(s_{t+1}|s_t) = \frac{\pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}}{\sum_{s_{t+1}}\pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}} \text{ and } \hat{\beta}(s_t) = \beta\sum_{s_{t+1}}\pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}.$$

and note that:

$$\begin{aligned}
\frac{\mathcal{R}_t V_{t+1}}{e_t(z^t)} &= \left(\sum_{s_{t+1}} \pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}\hat{V}_{t+1}^{1-\alpha}(s_{t+1})\right)^{1/1-\alpha} \\
&= \left(\sum_{s_{t+1}} \pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}\right)^{1/1-\alpha} \hat{\mathcal{R}}_t\hat{V}_{t+1}(s_{t+1})
\end{aligned}$$

Using the definition of $\hat{\beta}(s_t)$:

$$\hat{\beta}(s_t) = \beta\left(\sum_{s_{t+1}} \pi(s_{t+1}|s_t)\lambda(z_{t+1})^{1-\alpha}\right)^{\frac{1-\rho}{1-\alpha}},$$

we finally obtain the desired result:

$$\hat{V}_t(s^t) = \left[(1-\beta)\hat{c}_t^{1-\rho} + \hat{\beta}(s_t)(\hat{\mathcal{R}}_t\hat{V}_{t+1}(s^{t+1}))^{1-\rho}\right]^{\frac{1}{1-\rho}}$$

As before, if the z shocks are ii.d, then $\hat{\beta}$ is constant. ■

Proof of Proposition A.2:

Proof. First, we suppose the borrowing constraints are not binding, which is the easiest case. Assume the equilibrium allocations only depend on y^t, not on z^t. Then conditions 2.2 and 2.3 imply that the Euler equations of the Arrow economy, for the contingent claim and the stock respectively, read as follows:

$$\begin{aligned} 1 &= \frac{\hat{\beta}\hat{\phi}(z_{t+1})}{\hat{q}_t(z^t, z_{t+1})} \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \left(\frac{\hat{c}_{t+1}(y^t, y_{t+1})}{\hat{c}_t(y^t)}\right)^{-\rho} \left(\frac{\hat{V}_{t+1}(y^{t+1})}{\hat{V}_t(y^t)}\right)^{\rho-\alpha} \quad \forall z_{t+1} \\ 1 &= \hat{\beta} \sum_{z_{t+1}} \hat{\phi}(z_{t+1}) \left[\frac{\hat{v}_{t+1}(z^{t+1}) + \alpha}{\hat{v}_t(z^t)}\right] \qquad (59) \\ & * \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \left(\frac{\hat{c}_{t+1}(y^t, y_{t+1})}{\hat{c}_t(y^t)}\right)^{-\rho} \left(\frac{\hat{V}_{t+1}(y^{t+1})}{\hat{V}_t(y^t)}\right)^{\rho-\alpha} \quad all z_{t+1}. \qquad (60) \end{aligned}$$

In the first Euler equation, the only part that depends on z_{t+1} is $\frac{\hat{\phi}(z_{t+1})}{\hat{q}_t(z^t,z_{t+1})}$ which therefore implies that $\frac{\hat{\phi}(z_{t+1})}{\hat{q}_t(z^t,z_{t+1})}$ cannot depend on z_{t+1} : $\hat{q}_t(z^t, z_{t+1})$ is proportional to $\hat{\phi}(z_{t+1})$. Thus define $\hat{R}^A_t(z^t)$ by

$$\hat{q}_t(z^t, z_{t+1}) = \frac{\hat{\phi}(z_{t+1})}{\hat{R}^A_t(z^t)} \qquad (61)$$

as the risk-free interest rate in the stationary Arrow economy. Using this condition, the Euler equation in (38) simplifies to the following expression:

$$\begin{aligned} 1 &= \hat{\beta}\hat{R}^A_t(z^t) \sum_{y_{t+1}} \varphi(y_{t+1}|y_t) \left(\frac{\hat{c}_{t+1}(y^t, y_{t+1})}{\hat{c}_t(y^t)}\right)^{-\rho} \qquad (62) \\ & \left(\frac{\hat{V}_{t+1}(y^{t+1})}{\hat{V}_t(y^t)}\right)^{\rho-\alpha} \qquad (63) \end{aligned}$$

First, notice that apart from $\hat{R}^A_t(z^t)$ noting in this condition depends on z^t, so we can choose $\hat{R}^A_t(z^t) = \hat{R}^A_t$. ■